12.7 Strategies (which test to use on which series)

1) Check at a glance to see if $\lim_{n\to\infty} a_n \neq 0$.

If this is true, then the series diverges by the **Test for Divergence**.

2) Series that we can find whether or not they converge rather quickly:

$a)$ $\boxed{p-\text{series}}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent if $p \leq 1$.

$b)$ $\boxed{\text{geometric series}}$ $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^{n}$ is convergent for $|r| < 1$ and divergent if $|r| \geq 1$.

3) Use the Comparison Test / Limit Comparison Test on series

$a)$ that have the form of $p-$series or geometric series

if a_n is a fraction involving polynomials only or polynomials under radicals compare this series with a $p-$series

$b)$ Note: make sure that the series has only positive terms to use the comp. tests

12.7 Strategies (which test to use on which series)

4) The Alternating Series Test might work on series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$

5) The Ratio Test works well on series involving factorials
or constants raised to powers involving n

6) The Root Test works well if a_n is of the form $(b_n)^n$

7) The Integral Test works well if $\int_k^{\infty} f(x)\,dx$ is not difficult to evaluate,

where $\sum_{n=k}^{\infty} a_n$ with $a_n = f(n)$ and f is continuous, decreasing, and positive on $[k, \infty)$

Math 104 – Rimmer
12.7 Strategies to deal with all tests

$$i)\ \sum_{n=1}^{\infty}\frac{n}{n^3+2} \qquad iii)\ \sum_{n=1}^{\infty}\frac{n^2}{e^n} \qquad v)\ \sum_{n=1}^{\infty}\pi^{1/n} \qquad vii)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$$

$$ii)\ \sum_{n=1}^{\infty}\frac{n^2+2}{n^3+3} \qquad iv)\ \sum_{n=1}^{\infty}\frac{\cos(5n)}{1+(1.4)^n} \qquad vi)\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sqrt{n}}{n+1} \qquad viii)\ \sum_{n=2}^{\infty}\frac{n\ln n}{(n-1)^3}$$

$$i)\ \sum_{n=1}^{\infty}\frac{n}{n^3+2}$$

$$b_n=\frac{n}{n^3}=\frac{1}{n^2}$$

$$\frac{n}{n^3+2}<\frac{1}{n^2}\ \text{ since } n^3<n^3+2$$

$\sum_{n=1}^{\infty}b_n$ converges (p-series w/ $p=2$) so $\sum_{n=1}^{\infty}a_n$

converges by the Comp. Test.

$$ii)\ \sum_{n=1}^{\infty}\frac{n^2+2}{n^3+3}$$

$$b_n=\frac{n^2}{n^3}=\frac{1}{n}$$

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{n^2+2}{n^3+3}}{\frac{1}{n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n^3+2n}{n^3+3}\right|=1$$

$\sum_{n=1}^{\infty}b_n$ diverges (Harmonic series) so $\sum_{n=1}^{\infty}a_n$

diverges by the Limit Comp. Test.

$$iii)\ \sum_{n=1}^{\infty}\frac{n^2}{e^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\,\frac{e^n}{e^{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\underbrace{\frac{(n+1)^2}{n^2}}_{\to 1}\cdot\frac{e^n}{e^n\,e}\right|=\frac{1}{e}<1$$

so $\sum_{n=1}^{\infty}a_n$ converges

by the Ratio Test.

$i)\ \sum_{n=1}^{\infty} \frac{n}{n^3+2}$ $\quad iii)\ \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ $\quad v)\ \sum_{n=1}^{\infty} \pi^{1/n}$ $\quad vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$ii)\ \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3}$ $\quad iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ $\quad vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$ $\quad viii)\ \sum_{n=2}^{\infty} \frac{n \ln n}{(n-1)^3}$

$iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ terms aren't all positive

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\cos(5n)|}{1+(1.4)^n}$ now they are

$$b_n = \frac{1}{(1.4)^n} = \frac{1}{\left(\frac{7}{5}\right)^n} = \left(\frac{5}{7}\right)^n$$

$\sum_{n=1}^{\infty} b_n$ converges geom. series w/ $r=5/7$

$$\frac{|\cos(5n)|}{1+(1.4)^n} < \frac{1}{(1.4)^n}$$

so $\sum_{n=1}^{\infty} a_n$ converges absolutely by the Comp. Test.

$v)\ \sum_{n=1}^{\infty} \pi^{1/n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \pi^{1/n} = \pi^0 = 1 \neq 0$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by the Test For Divergence

$vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$

$$b_n = \frac{\sqrt{n}}{n+1} > 0$$

b_n is decreasing

$$\lim_{n\to\infty} b_n = 0$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test

$i)\ \sum_{n=1}^{\infty} \frac{n}{n^3+2}$ $\quad iii)\ \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ $\quad v)\ \sum_{n=1}^{\infty} \pi^{1/n}$ $\quad vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$ii)\ \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3}$ $\quad iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ $\quad vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$ $\quad viii)\ \sum_{n=2}^{\infty} \frac{n\ln n}{(n-1)^3}$

$vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ $\quad f(x) = \frac{\ln x}{x^2} > 0,$ continuous, decreasing

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-\ln x}{x}\right|_2^b + \int_2^b \frac{1}{x^2}\,dx$$

$u = \ln x \quad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \quad v = -\frac{1}{x}$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} + \frac{\ln 2}{2} + \frac{-1}{b} + \frac{1}{2}$$

$= \frac{1+\ln 2}{2}$ since $\lim_{b\to\infty} \frac{-\ln b}{b} \overset{L'H}{=} \lim_{b\to\infty} \frac{-\frac{1}{b}}{1} = 0$

and $\lim_{b\to\infty} \frac{-1}{b} = 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

by the Integral Test

$viii)\ \sum_{n=2}^{\infty} \frac{n\ln n}{(n-1)^3}$ $\quad b_n = \frac{n\ln n}{n^3} = \frac{\ln n}{n^2}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n\ln n}{(n-1)^3}}{\frac{\ln n}{n^2}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n\ln n}{(n-1)^3} \cdot \frac{n^2}{\ln n}\right| = \lim_{n\to\infty} \left|\frac{n^3}{(n-1)^3}\right| = 1$$

$\sum_{n=1}^{\infty} b_n$ converges so $\sum_{n=1}^{\infty} a_n$

converges by the Limit Comp. Test.