12.8 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where:

$a)$ x is a variable

$b)$ The c_n's are constants called the coefficients of the series.

For each fixed x, the series above is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of x and diverge for other values of x.

The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

whose domain is the set of all x for which the series converges.

$f(x)$ is reminiscent of a polynomial but it has infinitely many terms

If all c_n's $= 1$, we have

$$f(x) = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n$$

This is the geometric series with $r = x$.

The power series will converge for $|x| < 1$ and diverge for all other x.

$$a = 1, r = x \Rightarrow s = \frac{a}{1-r} = \frac{1}{1-x} \qquad \boxed{\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n}$$

In general, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is called a power series centered at a or a power series about a

We use the Ratio Test (or the Root Test) to find for what values of x the series converges.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ for convergence

R is called the **radius of convergence** (R.O.C.).

solve for $|x-a|$ to get $|x-a| < R$

$\Rightarrow -R < x-a < R$

$\Rightarrow a-R < x < a+R$

This is called the **interval of convergence** (I.O.C.).

Plug in the endpoints to check for convergence or divergence at the endpoints.

use square brackets [or]

use parentheses (or)

Find the radius of convergence and the interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}\cdot(-1)}{\cancel{(-1)^n}}\cdot\frac{\cancel{(n+1)^2}}{\cancel{n^2}}\cdot\frac{\cancel{2^n}}{\cancel{2^n}\cdot 2}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right| = \left|\frac{-x}{2}\right|$$

$\nearrow 1$

For convergence, this limit needs to be less than 1

$$\left|\frac{-x}{2}\right| < 1 \Rightarrow \frac{1}{2}|x| < 1 \Rightarrow |x| < 2 \qquad \text{so, } -2 < x < 2$$

Now we need to solve this inequality for $|x|$.

This is the radius of convergence.

Plug in $x=2$ and $x=-2$ to see if there is conv. or div. at the endpoints.

$\underline{x=2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2 2^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n n^2$$

Diverges by the Test for Divergence since $\lim_{n\to\infty}(-1)^n n^2$ does not exist.

$\underline{x=-2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2 (-2)^n}{2^n} = \sum_{n=1}^{\infty}\frac{\overbrace{\left((-1)\cdot(-2)\right)^n}^{2^n} n^2}{2^n} = \sum_{n=1}^{\infty} n^2$$

Diverges by the Test for Divergence since $\lim_{n\to\infty} n^2 = \infty$.

Radius of convergence: $R = 2$
Interval of convergence: $(-2,2)$

Find the radius of convergence and the interval of convergence.

Math 104 – Rimmer
12.8 Power Series

$$\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{3^n}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{(x+4)^{n+1}}{(x+4)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{3^n}\cdot 3}{\cancel{3^n}}\cdot\underbrace{\cancel{\frac{\sqrt{n}}{\sqrt{n+1}}}}_{\nearrow 1}\cdot\frac{\cancel{(x+4)^n}(x+4)}{\cancel{(x+4)^n}}\right| = |3(x+4)|$$

For convergence, this limit needs to be less than 1

$$|3(x+4)|<1 \Rightarrow 3|x+4|<1 \Rightarrow |x+4|<\frac{1}{3}$$

Now we need to solve this inequality for $|x+4|$.

This is the radius of convergence.

$$\text{so, } -\frac{1}{3}<x+4<\frac{1}{3}$$
$$-\frac{1}{3}-4<x<\frac{1}{3}-4$$
$$\frac{-13}{3}<x<\frac{-11}{3}$$

Plug in $x=\frac{-13}{3}$ and $x=\frac{-11}{3}$ to see if there is conv. or div. at the endpoints.

$\underline{x=\frac{-13}{3}}$

$$\sum_{n=1}^{\infty}\frac{3^n\left(\frac{-13}{3}+4\right)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{3^n\left(\frac{-1}{3}\right)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

Converges by the Alt. Series Test

$b_n=\frac{1}{\sqrt{n}}$ is decreasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$

$\underline{x=\frac{-11}{3}}$

$$\sum_{n=1}^{\infty}\frac{3^n\left(\frac{-11}{3}+4\right)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

Divergent p-series with $p=\frac{1}{2}$.

$$\boxed{\begin{array}{l}\text{R.O.C.: } R=\frac{1}{3}\\ \text{I.O.C. : } \left[\frac{-13}{3},\frac{-11}{3}\right)\end{array}}$$

Find the radius of convergence and the interval of convergence.

Math 104 – Rimmer
12.8 Power Series

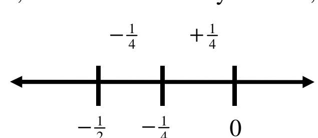
$$\sum_{n=1}^{\infty}\frac{(4x+1)^n}{n^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| = \lim_{n\to\infty}\left|\underbrace{\cancel{\frac{n^2}{(n+1)^2}}}_{\nearrow 1}\cdot\frac{\cancel{(4x+1)^n}\cdot(4x+1)}{\cancel{(4x+1)^n}}\right| = |4x+1|$$

For convergence, this limit needs to be less than 1

$|4x+1|<1$ For this one, the value a isn't very obvious, so we will proceed as follows:

$$\left|4\left(x+\tfrac{1}{4}\right)\right|<1$$
$$4\left|x+\tfrac{1}{4}\right|<1$$
$$\left|x+\tfrac{1}{4}\right|<\tfrac{1}{4}$$



it turns out $a=\frac{-1}{4}$

solve $4x+1=0$

so in this case with $a=\frac{-1}{4}$ and the interval going from $\frac{-1}{2}$ to 0,

the radius of convergence is $R=\frac{1}{4}$

Check endpoints:

$\underline{x=\frac{-1}{2}}$

$$\sum_{n=1}^{\infty}\frac{\left(4\left(\frac{-1}{2}\right)+1\right)^n}{n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

convergent Alt. series

$\underline{x=0}$

$$\sum_{n=1}^{\infty}\frac{(1)^n}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

convergent p-series

$$\boxed{\begin{array}{l}\text{R.O.C.: } R=\frac{1}{4}\\ \text{I.O.C. : } \left[\frac{-1}{2},0\right]\end{array}}$$

Sometimes the Root Test can be used just as the Ratio Test.

When a_n can be written as $(b_n)^n$, then the Root Test should be used.

$$\sum_{n=1}^{\infty} \frac{3^n (x-5)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{3(x-5)}{n} \right)^n$$

No value of x will make this limit >1 to give divergence

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left(\tfrac{3(x-5)}{n} \right)^n \right|} = \lim_{n\to\infty} \left| \frac{3(x-5)}{n} \right| = 0 < 1$$

We get convergence no matter what x is

$$\boxed{\begin{array}{l} \text{R.O.C. } = \infty \\ \text{I.O.C. } = (-\infty, \infty) \end{array}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \Rightarrow R.O.C. = \infty \Rightarrow I.O.C.\ (-\infty, \infty) \qquad \left(\text{or } \lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 \right)$$

the power series only converges for all x

$$\sum_{n=1}^{\infty} \frac{n!(x-7)^n}{2^n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{(x-7)^{n+1}}{(x-7)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2} \cdot \frac{\cancel{(x-7)^n} \cdot (x-7)}{\cancel{(x-7)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \tfrac{1}{2}(n+1)(x-7) \right| = \infty > 1$$

$$\boxed{\begin{array}{l} \text{R.O.C. } = 0 \\ \text{I.O.C. } = \{7\} \end{array}}$$

No value of x will make this limit <1 to give convergence

We get divergence for all values of x except at $x = a$

at $x = a$, each term of the series is 0

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \Rightarrow R.O.C. = 0 \Rightarrow I.O.C.\ \{a\} \qquad \left(\text{or } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty \right)$$

the power series only converges at the point $x = a$

Find the radius of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{\left[(n+1)!\right]^2}{(n!)^2}\cdot\frac{(2n)!}{\left[2(n+1)\right]!}\cdot\frac{x^{2(n+1)}}{x^{2n}}\right|$$

$\left[(n+1)n!\right]^2$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}(-1)}{\cancel{(-1)^n}}\cdot\frac{(n+1)^2\cancel{(n!)^2}}{\cancel{(n!)^2}}\cdot\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{x^{2n}}x^2}{\cancel{x^{2n}}}\right| = \lim_{n\to\infty}\left|(-1)x^2\cdot\cancel{\frac{n^2+2n+1}{4n^2+6n+2}}\right|$$

$\nearrow \frac{1}{4}$

$$= \left|\frac{(-1)x^2}{4}\right|$$

For convergence, this limit needs to be less than 1

$$\left|\frac{(-1)x^2}{4}\right| < 1 \Rightarrow \frac{1}{4}\left|x^2\right| < 1 \Rightarrow |x|^2 < 4 \Rightarrow |x| < 2$$

This is the radius of convergence.

$$\boxed{\text{Radius of convergence: } R = 2}$$