12.9 Functions as Power Series

The very first function we have seen represented as a power series is the geometric series with $a = 1$ and $r = x$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad , |x| < 1$$

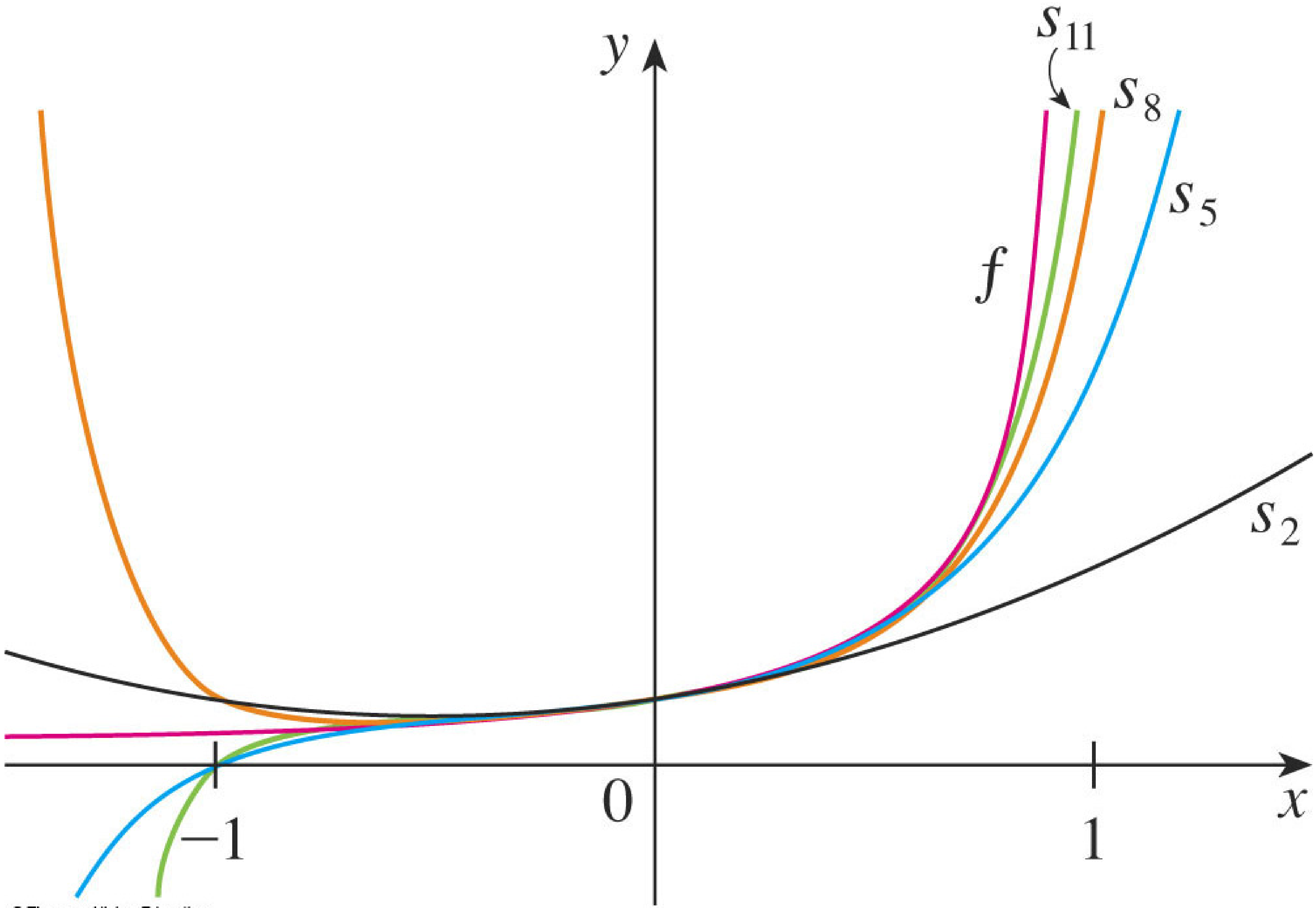
We can find the power series representation of other functions by algebraically manipulating them to to be some multiple of this series.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \quad , |x| < 1$$

The interval of convergence remains unchanged since this is still a type of geometric series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad , |x| < 1$$

$$= 1 - x + x^2 - x^3 + \cdots$$


y
s_{11}
s_8
s_5
f
s_2
0
−1
1
x

Represent the function as a power series and determine the interval of convergence.

$$\frac{3}{4+x^2}=3\left(\frac{1}{4+x^2}\right)=3\left(\frac{1}{4\left(1+\frac{x^2}{4}\right)}\right)=\frac{3}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right)=\frac{3}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right)$$

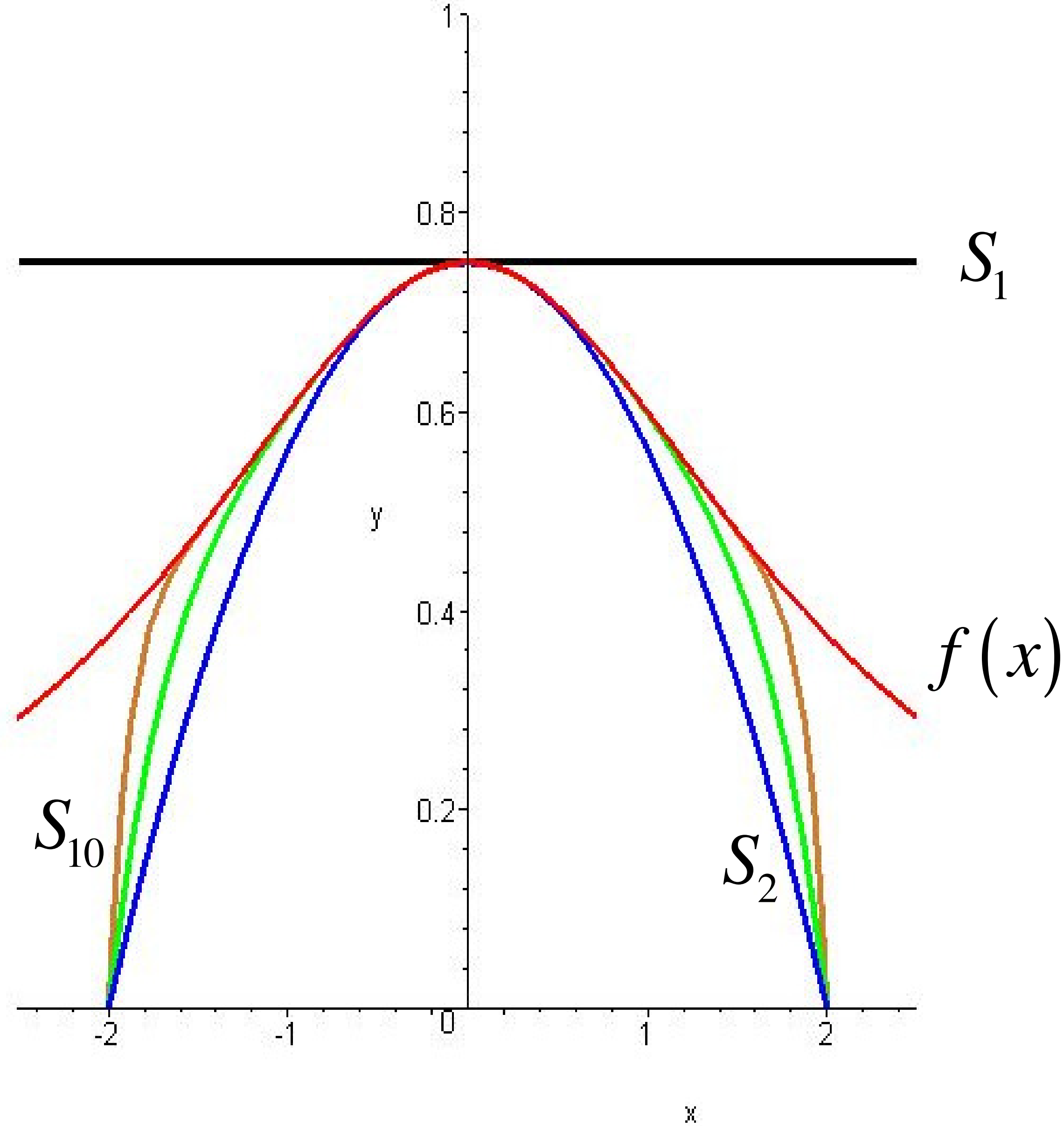
$$=\frac{3}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right)=\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{-x^2}{4}\right)^n$$

$$\left|\frac{-x^2}{4}\right|<1 \qquad \frac{1}{4}\left|x^2\right|<1 \Rightarrow |x|^2<4$$

, so the interval of convergence is $|x|<2$

$$\frac{3}{4+x^2}=\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n}, \quad |x|<2$$

$$=\frac{3}{4}-\frac{3}{16}x^2+\frac{3}{64}x^4-\frac{3}{128}x^6+\ldots$$


1
0.8
0.6
y
0.4
0.2
-2
-1
0
1
2
x
S_1
$f(x)$
S_{10}
S_2

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\cdots$$

If the power series representation of $f(x)$ has a radius of convergence $R>0$,

we can obtain a power series representation for $f'(x)$ by **term - by - term differentiation**:

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\cdots$$

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+\cdots$$

$$f'(x)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}c_n(x-a)^n\right)=\sum_{n=0}^{\infty}\frac{d}{dx}\left[c_n(x-a)^n\right]=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$$

starts at $n=1$

with the same radius of convergence R

we can obtain a power series representation for $\int f(x)\,dx$ by **term - by - term integration**:

$$\int f(x)\,dx=C+c_0x+c_1\frac{(x-a)^2}{2}+c_2\frac{(x-a)^3}{3}+c_3\frac{(x-a)^4}{4}+\cdots$$

$$\int\left(\sum_{n=0}^{\infty}c_n(x-a)^n\right)dx=\sum_{n=0}^{\infty}\int\left[c_n(x-a)^n\right]dx=C+\sum_{n=0}^{\infty}\frac{c_n(x-a)^{n+1}}{n+1}$$

with the same radius of convergence R

C is a constant of integration

Represent the function as a power series and determine the radius of convergence.

$$f(x)=\frac{x^3}{(1-x)^2}=x^3\left[\frac{1}{(1-x)^2}\right]$$

$$g(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad \text{with } R=1$$

$$g(x)=(1-x)^{-1}\Rightarrow g'(x)=-1(1-x)^{-2}(-1)$$

$$g'(x)=\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1} \quad \text{with } R=1$$

$$\frac{x^3}{(1-x)^2}=x^3g'(x)=x^3\sum_{n=1}^{\infty}nx^{n-1}=\sum_{n=1}^{\infty}nx^{n+2}$$

$$\frac{x^3}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n+2} \quad \text{with } R=1$$

Represent the function as a power series and determine the radius of convergence.

$$f(x) = \arctan x$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{with } R = 1$$

$$= 1 - x^2 + x^4 - x^6 + \ldots$$

$$\int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{, with } R = 1$$

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \arctan 0 = C + 0 = 0 \Rightarrow C = 0$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{,with } R = 1$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Represent the function as a power series and determine the radius of convergence.

$$f(x)=\ln(1-x)$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad \text{with } R=1$$

$$\int\frac{1}{1-x}\,dx=C+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$-\ln(1-x)=C+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \qquad -\ln(1-0)=C+0\Rightarrow C=0$$

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\text{ ,with } R=1$$

$$=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{,with } R = 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

$$\boxed{\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots}$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{,with } R = 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\ln\left(1 - \tfrac{1}{2}\right) = -\tfrac{1}{2} - \frac{\left(\tfrac{1}{2}\right)^2}{2} - \frac{\left(\tfrac{1}{2}\right)^3}{3} - \frac{\left(\tfrac{1}{2}\right)^4}{4} + \ldots$$

$$\ln\left(\tfrac{1}{2}\right) = -\tfrac{1}{2} - \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \ldots$$

$$\ln 1 - \ln 2 = -\tfrac{1}{2} - \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \ldots$$

$$\boxed{\ln 2 = \tfrac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \ldots}$$

Algebraically manipulate $\dfrac{1}{(1-x)^2}$ $\left(\text{the same way we manipulated } \dfrac{1}{1-x}\right)$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{with } R = 1$$

Represent $\dfrac{1}{(4-3x)^2}$ as a power series and determine the radius of convergence.

$$\frac{1}{(4-3x)^2} = \frac{1}{\left(4\left[1-\frac{3x}{4}\right]\right)^2} = \frac{1}{16}\cdot\frac{1}{\left(1-\frac{3x}{4}\right)^2}$$

$$= \frac{1}{16}\cdot\sum_{n=1}^{\infty} n\left(\tfrac{3x}{4}\right)^{n-1} = \frac{1}{4^2}\cdot\sum_{n=1}^{\infty}\frac{n3^{n-1}x^{n-1}}{4^{n-1}}$$

$$= \sum_{n=1}^{\infty}\frac{n3^{n-1}x^{n-1}}{4^{n-1+2}}$$

$$\frac{1}{\left(1-\frac{3x}{4}\right)^2} = \sum_{n=1}^{\infty} n\left(\tfrac{3x}{4}\right)^{n-1}$$

$$\left|\tfrac{3x}{4}\right| < 1 \Rightarrow \tfrac{3}{4}|x| < 1$$

$$|x| < \frac{4}{3}, \quad R = \frac{4}{3}$$

$$\frac{1}{(4-3x)^2} = \sum_{n=1}^{\infty}\frac{n3^{n-1}x^{n-1}}{4^{n+1}} \quad , R = \frac{4}{3}$$

Algebraically manipulate $\ln(1-x)$ $\left(\text{the same way we manipulated } \dfrac{1}{1-x}\right)$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \text{ ,with } R=1$$

Represent $\ln(3+2x)$ as a power series and determine the radius of convergence.

$$\ln(3+2x) = \ln\left(3\left[1+\tfrac{2x}{3}\right]\right) = \ln\left(3\left[1-\tfrac{-2x}{3}\right]\right)$$

$$= \ln 3 + \ln\left(1-\tfrac{-2x}{3}\right) \text{ since } \ln(ab) = \ln a + \ln b$$

$$= \ln 3 + \left[-\sum_{n=0}^{\infty} \frac{\left(\frac{-2x}{3}\right)^{n+1}}{n+1}\right]$$

$$(-1)(-1)^{n+1} = (-1)^{n+2} = (-1)^n$$

$$\ln\left(1-\tfrac{-2x}{3}\right) = -\sum_{n=0}^{\infty} \frac{\left(\frac{-2x}{3}\right)^{n+1}}{n+1}$$

$$\left|\tfrac{-2x}{3}\right| < 1 \Rightarrow \tfrac{2}{3}|x| < 1$$

$$|x| < \frac{3}{2}, \quad R = \frac{3}{2}$$

$$\ln(3+2x) = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{n+1}}{3^{n+1}(n+1)}, \quad R = \frac{3}{2}$$