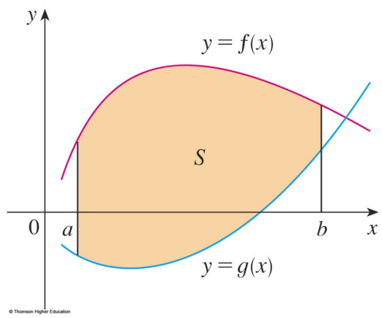
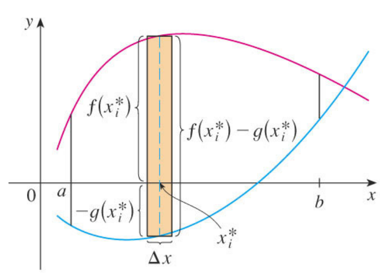
6.1 Area Between Curves

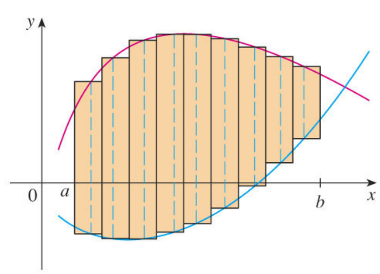


Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$.

Here, f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a,b]$.



(a) Typical rectangle



(b) Approximating rectangles

We divide S into n strips of equal width and approximate the i th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$

The Riemann sum $\sum_{i=1}^{n}\left[f(x_i^*)-g(x_i^*)\right]\Delta x$

is therefore an approximation to what
we intuitively think of as the area of S.

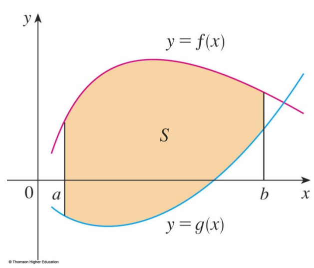
This approximation appears to become better and better as $n \to \infty$.

Thus, we define the area A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i^*)-g(x_i^*)\right]\Delta x$$

Thus, we have the following formula for area :

$$A = \int_a^b \left[f(x)-g(x)\right]dx$$



Remember S is described as the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where, f and g are continuous and $f(x) \geq g(x)$ for all x in $[a,b]$.

$$A = \text{the area of } S$$

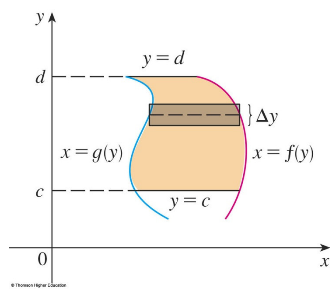
$$A = \int_a^b \left[\begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix}\right]dx$$

Some regions are best treated by regarding x as a function of y.

If a region is bouned by the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for all y in $[c, d]$, then its area is:

$$A = \int_c^d [f(y) - g(y)]\,dy$$

$$A = \int_c^d \left[\begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} \right] dy$$

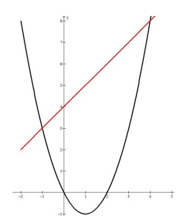


Find the area of the region bounded by the curves.

$y = x^2 - 2x,\ y = x + 4$

Select the correct answer.

a. $\frac{125}{3}$ b. $\frac{25}{3}$ c. 5 d. 20 e. $\frac{125}{6}$



Find where the curves intersect:

$$x^2 - 2x = x + 4$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$\Rightarrow x = 4, x = -1$$

$$\Rightarrow A = \int_{-1}^{4} \left[(x+4) - (x^2 - 2x) \right] dx = \int_{-1}^{4} \left[-x^2 + 3x + 4 \right] dx = \left[\frac{-x^3}{3} + \frac{3x^2}{2} + 4x \right]_{-1}^{4}$$

$$= \left[\frac{-4^3}{3} + \frac{3 \cdot 4^2}{2} + 4 \cdot 4 \right] - \left[\frac{-(-1)^3}{3} + \frac{3 \cdot (-1)^2}{2} + 4 \cdot (-1) \right]$$

$$= \left[\frac{-64}{3} + 24 + 16 \right] - \left[\frac{1}{3} + \frac{3}{2} - 4 \right]$$

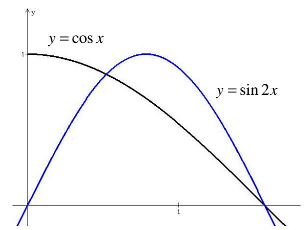
$$= \frac{-65}{3} + 44 - \frac{3}{2} = \frac{-130 + 264 - 9}{6} = \boxed{\frac{125}{6}}$$

Find the area of the region bounded by the curves.

$y = \cos x,\ y = \sin 2x,\ x = 0,\ x = \pi/2$

Select the correct answer.

a. $\frac{1}{4}$ b. $\frac{1}{2}$ c. $\frac{1}{3}$ d. 2 e. 4



Find where the curves intersect:

$\cos x = \sin 2x$

$\sin 2x - \cos x = 0$

Use the trig. ident. :

$\sin 2x = 2\sin x \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\Rightarrow x = \dfrac{\pi}{2}$ or $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}$

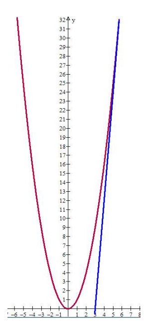
$$\Rightarrow A = \int_0^{\pi/6} [\cos x - \sin 2x]\,dx + \int_{\pi/6}^{\pi/2} [\sin 2x - \cos x]\,dx$$

$$= [\sin x + \tfrac{1}{2}\cos 2x]_0^{\pi/6} + [-\tfrac{1}{2}\cos 2x - \sin x]_{\pi/6}^{\pi/2}$$

$$= \left[\sin\left(\pi/6\right) + \tfrac{1}{2}\cos\left(\pi/3\right)\right] - \left[\sin(0) + \tfrac{1}{2}\cos(0)\right] + \left[-\tfrac{1}{2}\cos(\pi) - \sin\left(\pi/2\right)\right] - \left[-\tfrac{1}{2}\cos\left(\pi/3\right) - \sin\left(\pi/6\right)\right]$$

$$= [\tfrac{1}{2} + \tfrac{1}{4}] - [0 + \tfrac{1}{2}] + [\tfrac{1}{2} - 1] - [-\tfrac{1}{4} - \tfrac{1}{2}] = \tfrac{1}{4} + \tfrac{1}{4} = \boxed{\frac{1}{2}}$$

Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at (6, 36), and the x-axis.



We need the equation of the tangent line

$y = x^2 \Rightarrow y' = 2x$ evaluated at $x = 6 \Rightarrow y' = 12$

this is the slope of the tangent line. The tangency pt. is $(6, 36)$

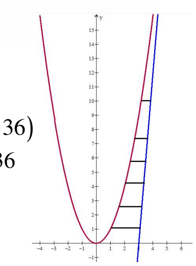
using $y = mx + b \Rightarrow 36 = 12(6) + b \Rightarrow 36 = 72 + b \Rightarrow b = -36$

$\Rightarrow$ the equation of the line is $y = 12x - 36$

$\Rightarrow$ the integral should be done in terms of dy

since the lower limit will change when you reach 3

$\Rightarrow$ the curves need to be expressed as functions of y



$y = x^2 \Rightarrow x = \sqrt{y}$

$y = 12x - 36 \Rightarrow x = \dfrac{y + 36}{12} \Rightarrow x = \dfrac{y}{12} + 3$

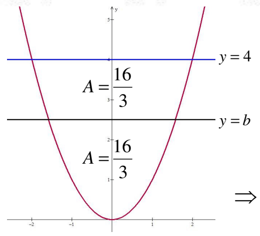
$$\Rightarrow A = \int_0^{36} \left[\left(\frac{y}{12} + 3\right) - \left(\sqrt{y}\right)\right] dy = \left(\frac{y^2}{24} + 3y - \frac{2y^{3/2}}{3}\right)_0^{36}$$

$$= \left(\frac{36^2}{24} + 3\cdot 36 - \frac{2\cdot 36^{3/2}}{3}\right) = 36\left(\frac{36}{24} + 3 - \frac{2\cdot 36^{1/2}}{3}\right) = 36\left(\frac{3}{2} + 3 - 4\right) = 36\left(\frac{1}{2}\right) = \boxed{18}$$

Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Select the correct answer.

a. $4^{2/3}$ b. $4^{1/3}$ c. $1/3$ d. $2^{1/2}$ e. $2/3$



Area b/w $y = 4$ and $y = x^2$

$$\Rightarrow A = \int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= 2\int_{0}^{2} \left(4 - x^2\right) dx \quad (\text{since it is even})$$

$$= 2\left(4x - \frac{x^3}{3}\right)_0^2 = 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$$

$\Rightarrow$ Area b/w $y = b$ and $y = x^2$ equals $\frac{16}{3}$ $\quad \Rightarrow A = \int_{-\sqrt{b}}^{\sqrt{b}} \left(b - x^2\right) dx = \frac{16}{3}$

$$\Rightarrow 2\int_{0}^{\sqrt{b}} \left(b - x^2\right) dx = \frac{16}{3} \quad \Rightarrow \left(bx - \frac{x^3}{3}\right)_0^{\sqrt{b}} = \frac{8}{3} \quad \Rightarrow \left(b^{3/2} - \frac{b^{3/2}}{3}\right) = \frac{8}{3}$$

(since it is even)

$$\Rightarrow b^{3/2}\left(1 - \frac{1}{3}\right) = \frac{8}{3} \quad \Rightarrow b^{3/2}\left(\frac{2}{3}\right) = \frac{8}{3} \quad \Rightarrow b^{3/2} = \frac{8}{3} \cdot \frac{3}{2} \quad \Rightarrow b^{3/2} = 4$$

$$\Rightarrow \boxed{b = 4^{2/3}}$$