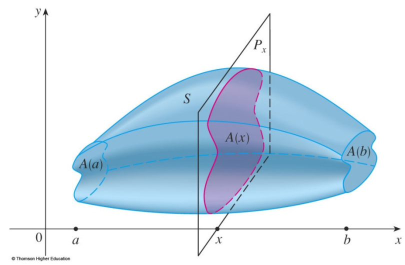
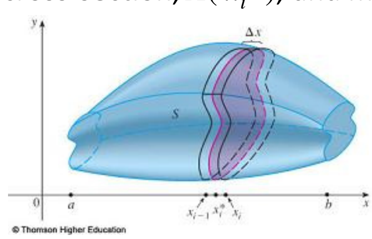


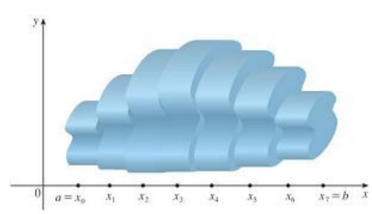
Goal: To find the volume of a solid

Method: "Cutting" the solid into many "pieces", find the volume of the pieces and add to find the total volume.

- The "pieces" are treated are cylinders.
- The base of each cylinder is called a **cross-section**.

- The volume of each cylinder is found by taking the area of the cross-section, $A(\,x_i^*\,)$, and multiplying by the height, Δx.





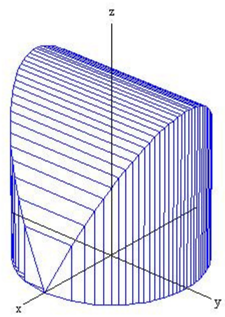
- The volume of the solid can be approximated by the sum of all cylinders.

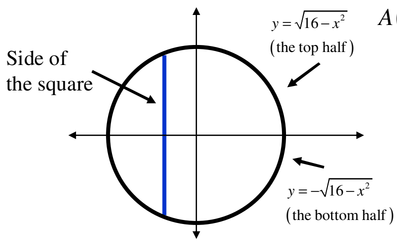
$$V \approx \sum_{i=1}^{n} A\left(x_i^*\right)\Delta x$$

- Taking the limit as the number of cylinders goes to infinity gives the exact volume of the solid.

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A\left(x_i^*\right)\Delta x \qquad \Rightarrow V = \int_a^b A(x)\,dx$$

A solid has a circular base of radius 4. If every plane cross section perpendicular to the x-axis is a square, then find the volume of the solid.





The equation of the circle is $x^2+y^2=16$.

The integral is done in terms of x
since the squares are moved horizontally.

$x^2+y^2=16 \Rightarrow y^2=16-x^2 \;\Rightarrow y=\pm\sqrt{16-x^2}$

The cross-sections are squares.

The length of the side of a square is $2\sqrt{16-x^2}$.

The area of a cross-section is :

$A(x)=\left(2\sqrt{16-x^2}\right)^2 \Rightarrow A(x)=4\left(16-x^2\right)$

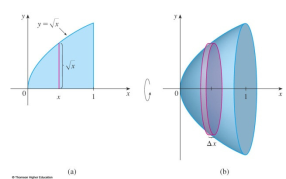
$$V=\int_a^b A(x)\,dx=\int_{-4}^{4}4\left(16-x^2\right)dx=\underbrace{2\int_0^4 4\left(16-x^2\right)dx}_{\text{since the integrand is even}}$$

$$=8\left(16x-\tfrac{x^3}{3}\right)\Big|_0^4=8\left(64-\tfrac{64}{3}\right)=8\cdot 64\left(1-\tfrac{1}{3}\right)$$

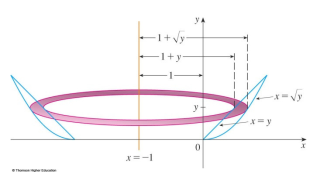
$$=8\cdot 64\left(\tfrac{2}{3}\right)=\boxed{\frac{1024}{3}}$$

When you revolve a plane region about an axis, the cross-sections are circular and the solid generated is called a **solid of revolution**.

If there is **no gap** between the axis of rotation and the region, then the method used is called the **disk method**.



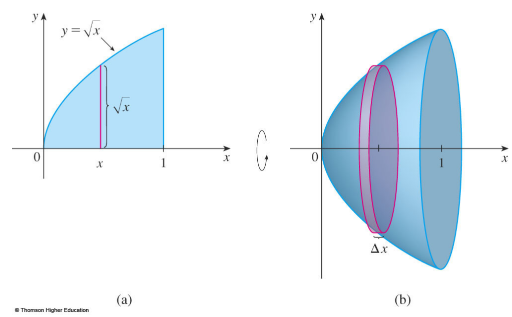
If there is **a gap** between the axis of rotation and the region, then the method used is called the **washer method**.



Disk Method with **horizontal axis of rotation** (not necessarily the x-axis)

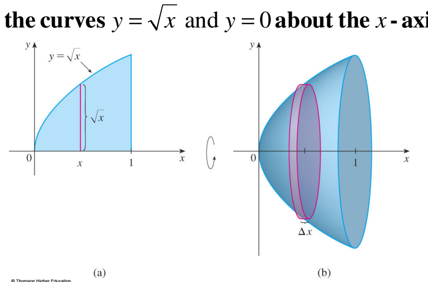
Cross-sections are circular: $A(x)=\pi\big[\underbrace{r(x)}_{\text{radius as a function of } x}\big]^2$

$$Volume=\int_a^b A(x)\,dx=\pi\int_a^b \big[\underbrace{r(x)}_{\text{radius as a function of } x}\big]^2\,dx$$



(a) (b)

Calculate the volume of the solid generated by rotating the region between the curves $y=\sqrt{x}$ and $y=0$ **about the** x**-axis**.

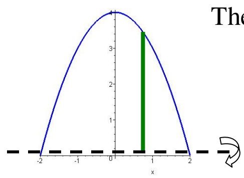


(a) (b)

$A(x)=\pi\big[\underbrace{r(x)}_{(\sqrt{x})^2}\big]^2$

$V=\pi\int_0^1 x\,dx=\pi\left[\frac{x^2}{2}\right]_0^1=\boxed{\frac{\pi}{2}\text{ un.}^3}$

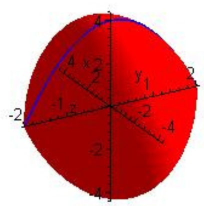
Calculate the volume of the solid generated by rotating the region between the curves $y = 4 - x^2$ and $y = 0$ **about the** x**-axis**.



The radius in this case is the distance from the x-axis to the function

$$r(x) = 4 - x^2$$

$$Volume = \int_a^b A(x)\,dx = \pi\int_a^b \left[r(x)\right]^2 dx$$



$$V = \pi\int_{-2}^{2}\left[4 - x^2\right]^2 dx = \pi\int_{-2}^{2}\left[16 - 8x^2 + x^4\right]dx$$

$$= 2\pi\int_0^2\left[16 - 8x^2 + x^4\right]dx = 2\pi\left[16x - \tfrac{8x^3}{3} + \tfrac{x^5}{5}\right]_0^2$$

since the integrand is even

$$= 2\pi\left[32 - \tfrac{64}{3} + \tfrac{32}{5}\right] = 2\pi\cdot 32\left[1 - \tfrac{2}{3} + \tfrac{1}{5}\right] = 64\pi\left[\frac{15 - 10 + 3}{15}\right]$$

$$= 64\pi\left[\frac{8}{15}\right] = \boxed{\frac{512\pi}{15}}$$

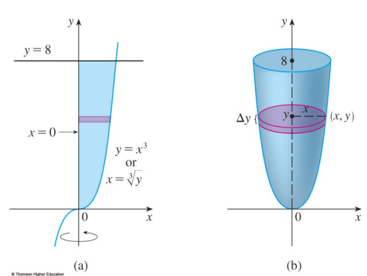
Disk Method with **vertical axis of rotation** (not necessarily the y-axis)

Cross-sections are circular: $$A(y) = \pi\left[r(y)\right]^2$$

radius as a function of y

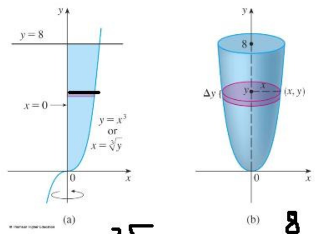
$$Volume = \int_a^b A(y)\,dy = \pi\int_a^b \left[r(y)\right]^2 dy$$

radius as a function of y



(a) (b)

Calculate the volume of the solid generated by rotating the region between the curves $y = x^3$, $y = 8$, and $x = 0$ **about the** y **- axis.**



$\left(\underbrace{8^{1/3}}_{2}\right)^5$

$\underbrace{x=\sqrt[3]{y}}_{r(y)}$

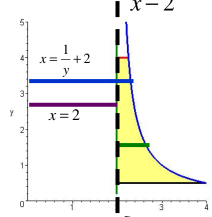
$V = \int_0^8 \pi \left[\sqrt[3]{y}\right]^2 dy$ ($r(y)^2$)

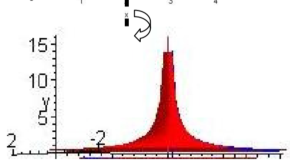
$= \pi \int_0^8 y^{2/3}\,dy = \pi \frac{3}{5} y^{5/3} \Big]_0^8$

$= \frac{3\pi}{5} \cdot \left(8^{5/3} - 0\right)$

$32 \cdot \frac{3\pi}{5} = \boxed{\frac{96\pi}{5}}$

Calculate the volume of the solid generated by rotating the region between the curves $y = \frac{1}{x-2}$ and $x = 2$, $y = \frac{1}{2}$, and $y = 4$ **about the line** $x = 2$.





The radius in this case is the horizontal distance from $x = 2$ to the curve $y = \frac{1}{x-2}$

We need to solve for x in terms of y:

$$y = \frac{1}{x-2} \Rightarrow \underbrace{\frac{y}{1} = \frac{1}{x-2}}_{\text{cross multiply}} \Rightarrow y(x-2) = 1 \Rightarrow x - 2 = \frac{1}{y} \Rightarrow x = \frac{1}{y} + 2$$

The radius is then:

$$r(y) = \left(\frac{1}{y} + 2\right) - 2 \Rightarrow r(y) = \frac{1}{y}$$

$$Volume = \int_a^b A(y)\,dx = \pi \int_a^b \left[r(y)\right]^2 dy = \pi \int_{1/2}^{4} \left(\frac{1}{y}\right)^2 dy = \pi \int_{1/2}^{4} y^{-2} dy$$

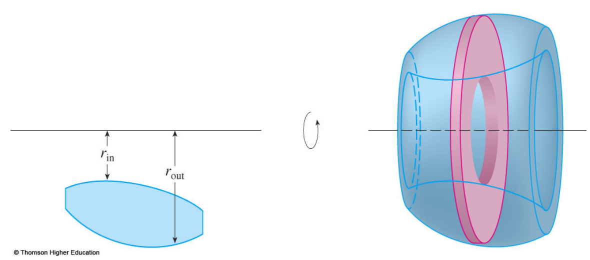
$$= \pi \left[\frac{-1}{y}\right]_{1/2}^{4} = \pi \left[\frac{-1}{4} + 2\right] = \boxed{\frac{7\pi}{4}}$$

Washer Method with **horizontal axis of rotation** (not necessarily the x-axis)

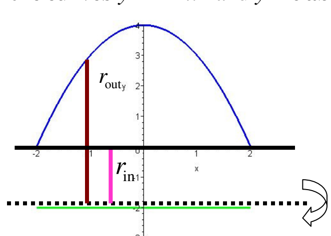
Draw a radius from the axis of rotation to the outer curve and call this **outer radius**

Draw a radius from the axis of rotation to the inner curve and call this **inner radius**

$$Volume=\int_a^b A(x)\,dx \;=\pi\int_a^b\left(\underbrace{\left[r_{out}(x)\right]^2}_{\text{outer radius as a function of } x}-\underbrace{\left[r_{in}(x)\right]^2}_{\text{inner radius as a function of } x}\right)dx$$



Calculate the volume of the solid generated by rotating the region between the curves $y=4-x^2$ and $y=0$ **about the line** $y=-2$.

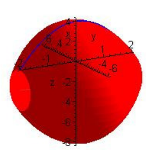


The outside radius (r_{out}) can be found by drawing a line from the axis of rotation **through** the region.

$$r_{\text{out}}=2+\left(4-x^2\right)$$
$$r_{\text{out}}=6-x^2$$

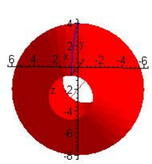
The inside radius (r_{in}) can be found by drawing a line from the axis of rotation **to** the region.

$$r_{\text{in}}=2$$



$$V=\pi\int_a^b\left(\underbrace{\left[r_{out}(x)\right]^2}_{\text{outer radius as a function of } x}-\underbrace{\left[r_{in}(x)\right]^2}_{\text{inner radius as a function of } x}\right)dx=\pi\int_{-2}^{2}\left(\left[6-x^2\right]^2-\left[2\right]^2\right)dx$$

$$=\pi\int_{-2}^{2}\left(\left(36-12x^2+x^4\right)-4\right)dx\;=\pi\int_{-2}^{2}\left(32-12x^2+x^4\right)dx$$



$$=2\pi\int_0^2\left(32-12x^2+x^4\right)dx\;=2\pi\left[32x-4x^3+\tfrac{x^5}{5}\right]_0^2$$

since the integrand is even

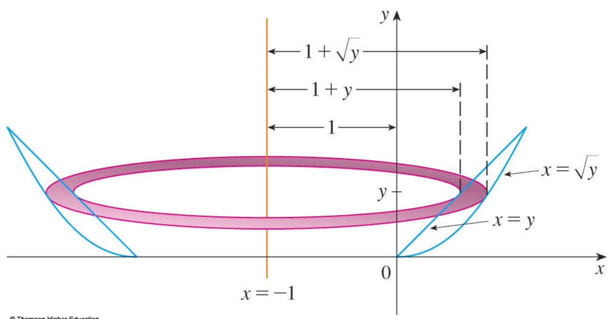
$$=2\pi\left[64-32+\tfrac{32}{5}\right]=2\pi\cdot 32\left[2-1+\tfrac{1}{5}\right]=64\pi\left[\tfrac{6}{5}\right]=\boxed{\frac{384\pi}{5}}$$

Washer Method with **vertical axis of rotation** (not necessarily the y-axis)

Draw a radius from the axis of rotation to the outer curve and call this **outer radius**

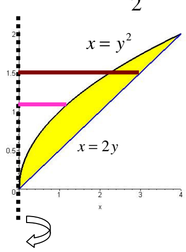
Draw a radius from the axis of rotation to the inner curve and call this **inner radius**

$$Volume = \int_a^b A(y)\,dy = \pi \int_a^b \left(\underbrace{\left[r_{out}(y) \right]^2}_{\text{outer radius as a function of } y} - \underbrace{\left[r_{in}(y) \right]^2}_{\text{inner radius as a function of } y} \right) dy$$



Math 104 – Rimmer
6.2 Volumes

Calculate the volume of the solid generated by rotating the region between the curves $y = \frac{x}{2}$ and $y = \sqrt{x}$ **about the** y**-axis.**



We need to solve for x in terms of y:

$$r_{\text{out}} = 2y \qquad y = \frac{x}{2} \Rightarrow x = 2y$$

$$r_{\text{in}} = y^2 \qquad y = \sqrt{x} \Rightarrow x = y^2$$

$$V = \pi \int_a^b \left(\underbrace{\left[r_{out}(y) \right]^2}_{\text{outer radius as a function of } y} - \underbrace{\left[r_{in}(y) \right]^2}_{\text{inner radius as a function of } y} \right) dy$$

$\frac{x}{2} = \frac{\sqrt{x}}{1}$

$x = 2\sqrt{x}$

$x^2 = 4x$

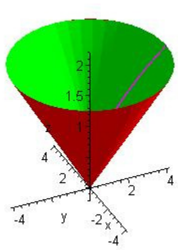
$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0 \qquad x = 4$

$\Downarrow \qquad \Downarrow$

$y = 0 \qquad y = 2$



$$V = \pi \int_0^2 \left([2y]^2 - \left[y^2 \right]^2 \right) dy = \pi \int_0^2 \left(4y^2 - y^4 \right) dy$$

$$= \pi \left[\frac{4y^3}{3} - \frac{y^5}{5} \right]_0^2 = \pi \left[\tfrac{32}{3} - \tfrac{32}{5} \right] = 32\pi \left[\tfrac{1}{3} - \tfrac{1}{5} \right]$$

$$= 32\pi \left[\tfrac{5-3}{15} \right] = \boxed{\frac{64\pi}{15}}$$