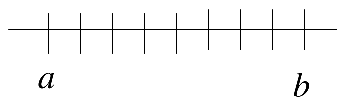
Average of $n$ numbers:

$$\frac{x_1 + x_2 + x_3 + \cdots x_n}{n}$$

Take the $n$ numbers to be sample points for a function:

$$\frac{f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + \cdots f\left(x_n^*\right)}{n}$$

Partition the interval $[a,b]$ into $n$ subintervals of equal length.




Here $n = 8$.

What is the width of each subinterval? $\frac{b-a}{n}$ Call this $\Delta x$

$$\Delta x = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{\Delta x}$$

We now have:

$$\frac{f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + \cdots f\left(x_n^*\right)}{\frac{b-a}{\Delta x}} = \frac{\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x}{b-a}$$



Taking the limit as $n \to \infty$, we get $\dfrac{\lim\limits_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x}{b-a}$

**Average value of** $f(x)$ on an interval $[a,b]$ $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Find the average value of the function $\dfrac{3}{(1+r)^2}$ on the interval $[1,6]$.

Average value of $f(x) = \dfrac{3}{(1+r)^2}$ on $[1,6]$ $= \dfrac{1}{5}\displaystyle\int_1^6 \frac{3}{(1+r)^2}\,dr$

$u = 1 + r \quad r = 1 \Rightarrow u = 2$

$du = dr \quad r = 6 \Rightarrow u = 7$

$$= \frac{1}{5}\int_2^7 \frac{3}{u^2}\,du = \frac{3}{5}\int_2^7 u^{-2}\,du = \frac{3}{5}\left[\frac{-1}{u}\right]_2^7 = \frac{3}{5}\left[\frac{-1}{7} - \frac{-1}{2}\right]$$

$$= \frac{3}{5}\left[\frac{-2}{14} + \frac{7}{14}\right] = \frac{3}{5}\left[\frac{5}{14}\right] = \boxed{\frac{3}{14}}$$

# Mean Value Theorem for Integrals

Let $f$ be continuous on $[a,b]$,

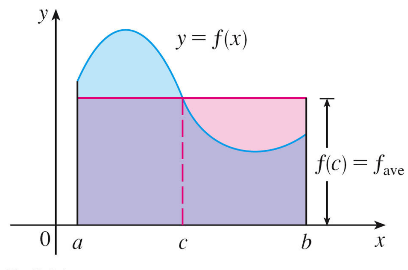
then there is a value $c$ in $[a,b]$ such that

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

*or*

$$f(c)(b-a)=\int_a^b f(x)\,dx$$

area of the rectangle = area under the curve




Math 104 – Rimmer
6.5 Average Value of a Function

Proof:

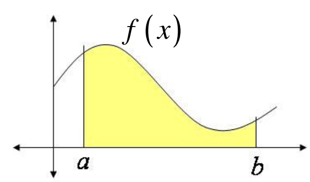
Let $f$ be continuous on $[a,b]$,

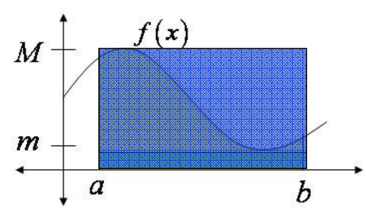
By the Extreme Value Theorem,

there is a $m$ and $M$ such that $m \le f(x) \le M$.

Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$




By the Intermediate Value Theorem,

there is a $c$ in $[a,b]$ with $f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$.

(a) Find the average value of $f(x)=\sqrt{x}$ on the interval $[0,4]$.

(b) Find $c$ such that $f_{ave}=f(c)$.

(c) Sketch the graph of $f$ and a rectangle whose area is the same as the area under the graph of $f$.

**(a)** $\begin{matrix}\text{Average value}\\ \text{of } f(x) \text{ on } [a,b]\end{matrix}=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \Rightarrow \begin{matrix}\text{Average value}\\ \text{of } f(x)=\sqrt{x} \text{ on } [0,4]\end{matrix}=\dfrac{1}{4}\displaystyle\int_0^4 x^{1/2}dx$

$$=\frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4=\frac{1}{6}\left(4^{3/2}\right)$$

$$=\frac{1}{6}\left(4^{1/2}\right)^3=\frac{8}{6}=\boxed{\frac{4}{3}}$$

**(b)** $f(c)=\dfrac{4}{3}\Rightarrow\sqrt{c}=\dfrac{4}{3}\Rightarrow c=\boxed{\dfrac{16}{9}}$

**(c)**