Section 8.1 Integration By Parts

Goal: To be able to integrate more functions.

Chapter 8: Techniques of Integration

- **8.1**: Integration By Parts
- **8.2**: Integrating Powers of Trig. Functions
- **8.3**: Trig. Substitution
- **8.4**: Partial Fraction Decomposition

Integration using **substitution** can be thought of as the **chain rule** in reverse.

Integration by parts can be thought of as the **product rule** in reverse.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f'(x)\cdot g(x)+f(x)\cdot g'(x)$$

$$\int \frac{d}{dx}\left[f(x)\cdot g(x)\right]dx=\int\left[f'(x)\cdot g(x)\right]dx+\int\left[f(x)\cdot g'(x)\right]dx$$

$$f(x)\cdot g(x)=\int\left[f'(x)\cdot g(x)\right]dx+\int\left[f(x)\cdot g'(x)\right]dx$$

$$f(x)\cdot g(x)-\int\left[f'(x)\cdot g(x)\right]dx=\int\left[f(x)\cdot g'(x)\right]dx$$

$$\int\left[f(x)\cdot g'(x)\right]dx=f(x)\cdot g(x)-\int\left[f'(x)\cdot g(x)\right]dx$$

$$\int \overset{u}{f(x)} \cdot \overset{dv}{g'(x)\,dx} = \overset{u}{f(x)} \cdot \overset{v}{g(x)} - \int \overset{v}{g(x)} \cdot \overset{du}{f'(x)\,dx}$$

$$u = f(x) \qquad v = g(x)$$
$$du = f'(x)\,dx \quad dv = g'(x)\,dx$$

$$\boxed{\int u\,dv = u \cdot v - \int v\,du}$$

Choose u and choose dv

Big Picture: We are trading in one integral for another

$\int u\,dv$ $\qquad$ $\int v\,du$

Goal: To get a **simpler** integral than the original one

1. Choose u to be a function that becomes simpler when **differentiated**
2. Make sure dv can be readily **integrated**

$$\int x^2 e^{5x}\,dx$$

Wrong Way

$u = e^{5x} \qquad dv = x^2 dx$

$du = 5e^{5x} dx \quad v = \dfrac{x^3}{3}$

traded in this $\qquad$ for this

$$\overbrace{\int x^2 e^{5x}\,dx} = \tfrac{1}{3} x^3 e^{5x} - \overbrace{\int \tfrac{5}{3} x^3 e^{5x}\,dx}$$

Correct Way

$u = x^2 \qquad dv = e^{5x} dx$

$du = 2x\,dx \quad v = \tfrac{1}{5} e^{5x}$

$$\overbrace{\int x^2 e^{5x}\,dx} = \tfrac{1}{5} x^2 e^{5x} - \overbrace{\int \tfrac{2}{5} x e^{5x}\,dx}$$

$$\int x^2 e^{5x}\,dx = \tfrac{1}{5} x^2 e^{5x} - \tfrac{2}{5} \boxed{\int x e^{5x}\,dx}$$

I.B.P. again

$u = x \qquad dv = e^{5x} dx$

$du = dx \quad v = \tfrac{1}{5} e^{5x}$

$$\int x^2 e^{5x}\,dx = \tfrac{1}{5} x^2 e^{5x} - \tfrac{2}{5} \overbrace{\left[\tfrac{1}{5} x e^{5x} - \tfrac{1}{5} \int e^{5x}\,dx \right]}$$

$$\int x^2 e^{5x}\,dx = \tfrac{1}{5} x^2 e^{5x} - \tfrac{2}{25} x e^{5x} + \tfrac{2}{25} \int e^{5x}\,dx$$

$$\boxed{\int x^2 e^{5x}\,dx = \tfrac{1}{5} x^2 e^{5x} - \tfrac{2}{25} x e^{5x} + \tfrac{2}{125} e^{5x} + C}$$

Shortcut: Works when you have one of the following two situations :

1. $\int (\text{polynomial})(\text{exponential})\,dx$
2. $\int (\text{polynomial})(\text{trig.})\,dx$

$$\int x^2 e^{5x} dx$$

Step 1 : Differentiate the polynomial down to 0.

Step 2 : Integrate the trig. or exponential the same amount of times

Step 3 : Multiply along diagonals going down to the right applying an alternating sign starting with +

Diff		*Int*
x^2	$+$	e^{5x}
$2x$	$-$	$\frac{1}{5}e^{5x}$
2	$+$	$\frac{1}{25}e^{5x}$
0		$\frac{1}{125}e^{5x}$

$$\boxed{\int x^2 e^{5x} dx = \tfrac{1}{5}x^2 e^{5x} - \tfrac{2}{25}xe^{5x} + \tfrac{2}{125}e^{5x} + C}$$

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8.1 Integration By Parts

$$\int_0^1 x\cos(\pi x)\,dx$$

Diff		*Int*
x	$+$	$\cos(\pi x)$
1	$-$	$\frac{1}{\pi}\sin(\pi x)$
0		$\frac{-1}{\pi^2}\cos(\pi x)$

$$\int_0^1 x\cos(\pi x)\,dx = \left[\frac{x}{\pi}\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x)\right]_0^1$$

$$= \left[\frac{1}{\pi}\sin(\pi) + \frac{1}{\pi^2}\cos(\pi)\right] - \left[0\sin 0 + \frac{1}{\pi^2}\cos(0)\right]$$

$$= \left[0 - \frac{1}{\pi^2}\right] - \left[0 + \frac{1}{\pi^2}\right] = \boxed{-\frac{2}{\pi^2}}$$

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy$$ ✖ **Short-cut doesn't work here**

Wrong Way

$u = \frac{1}{\sqrt{y}}$ $dv = \ln y\;dy$

$du = \frac{-1}{2y^{3/2}}dy$ $v = ????$

$f = y^{-1/2}$

$f' = -\frac{1}{2}y^{-3/2} \Rightarrow f' = \frac{-1}{2y^{3/2}}$

Correct Way

$u = \ln y$ $dv = \frac{1}{\sqrt{y}}\,dy$

$du = \frac{1}{y}dy$ $v = 2y^{1/2}$

$$\int y^{-1/2}dy = \frac{y^{1/2}}{1/2} = 2y^{1/2}$$

$$\int \frac{\ln y}{\sqrt{y}}dy = 2y^{1/2}\ln y - \int \frac{2y^{1/2}}{y}dy = 2y^{1/2}\ln y - 2\int y^{-1/2}dy = 2y^{1/2}\ln y - 2\cdot 2y^{1/2}$$

$$\int_4^9 \frac{\ln y}{\sqrt{y}}dy = \left[2\sqrt{y}\ln y - 4\sqrt{y}\right]_4^9 = \left[2\sqrt{9}\ln 9 - 4\sqrt{9}\right] - \left[2\sqrt{4}\ln 4 - 4\sqrt{4}\right]$$

$$= [6\ln 9 - 12] - [4\ln 4 - 8] = \boxed{6\ln 9 - 4\ln 4 - 4}$$

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8.1 Integration By Parts

$$\int_1^{\sqrt{3}} \arctan\left(1/x\right)dx$$ ✖ **Short-cut doesn't work here**

Only Way

$u = \arctan\left(1/x\right)$ $dv = dx$

$du = \frac{-1}{x^2+1}dx$ $v = x$

$f = \arctan\left(1/x\right)$

$f' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(\frac{-1}{x^2}\right) \Rightarrow f' = \frac{1}{1+\frac{1}{x^2}}\left(\frac{-1}{x^2}\right) \Rightarrow f' = \frac{-1}{x^2+1}$

U – substitution

$u = x^2+1$

$du = 2x\,dx$ $\frac{1}{2}du = x\,dx$

$\int \frac{x}{x^2+1}dx \Rightarrow \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln u \Rightarrow \frac{1}{2}\ln\left(x^2+1\right)$

$$\int \arctan\left(1/x\right)dx = x\arctan\left(1/x\right) + \int \frac{x}{x^2+1}dx$$

$$\int \arctan\left(1/x\right)dx = x\arctan\left(1/x\right) + \frac{1}{2}\ln\left(x^2+1\right)$$

$$\int_1^{\sqrt{3}} \arctan\left(1/x\right)dx = \left[x\arctan\left(1/x\right) + \frac{1}{2}\ln\left(x^2+1\right)\right]_1^{\sqrt{3}}$$

$$= \left[\sqrt{3}\arctan\left(1/\sqrt{3}\right) + \frac{1}{2}\ln(4)\right] - \left[\arctan(1) + \frac{1}{2}\ln(2)\right]$$

$$= \sqrt{3}\frac{\pi}{6} + \frac{1}{2}(\ln 4 - \ln 2) - \frac{\pi}{4} = \boxed{\frac{\pi\sqrt{3}}{6} + \frac{1}{2}\ln 2 - \frac{\pi}{4}}$$

$\int e^{-t}\sin(t)\,dt$ ✖ **Short-cut doesn't work here**

$u=\sin(t) \quad dv=e^{-t}dt$

$du=\cos(t)dt \quad v=-e^{-t}$

$$\Rightarrow \int e^{-t}\sin(t)\,dt = -e^{-t}\sin(t)+\underbrace{\int e^{-t}\cos(t)\,dt}$$

$u=\cos(t) \quad dv=e^{-t}dt$

$du=-\sin(t)dt \quad v=-e^{-t}$

$$-e^{-t}\cos(t)-\int e^{-t}\sin(t)\,dt$$

$$\Rightarrow \int e^{-t}\sin(t)\,dt = -e^{-t}\sin(t)-e^{-t}\cos(t)-\int e^{-t}\sin(t)\,dt$$

$$+\int e^{-t}\sin(t)\,dt \qquad\qquad +\int e^{-t}\sin(t)\,dt$$

$$\Rightarrow 2\int e^{-t}\sin(t)\,dt = -e^{-t}\sin(t)-e^{-t}\cos(t)$$

$$\Rightarrow \int e^{-t}\sin(t)\,dt = \frac{1}{2}\left[-e^{-t}\sin(t)-e^{-t}\cos(t)\right]+C$$

Hierarchy Acronym to aid in choosing u

L : logarithmic functions

I : inverse trigonometric functions

P : polynomial functions

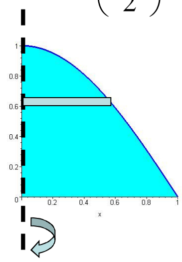
T : trigonometric functions

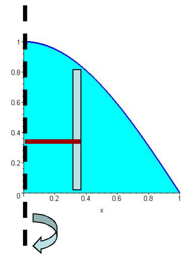
E : exponential functions

(T and E are interchangeable)

Find the volume of the solid of revolution formed by rotating the region bounded by the curves

$y = \cos\left(\frac{\pi x}{2}\right)$, $y = 0$, $0 \le x \le 1$ about the $y-$axis.





Try disk method

Problem: need to solve for x in terms of y

$$x = \frac{2}{\pi}\cos^{-1} y$$

Use Shells

radius: x

height: $\cos\left(\frac{\pi x}{2}\right)$

$$V = \int_a^b 2\pi(\text{radius})(\text{height})\,dx$$

$$V = 2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx$$

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$$V = 2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx$$

Diff		*Int*
x	$+$	$\cos\left(\frac{\pi x}{2}\right)$
1	$-$	$\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)$
0		$\frac{-4}{\pi^2}\cos\left(\frac{\pi x}{2}\right)$

$$V = 2\pi\left[\frac{2x}{\pi}\sin\left(\frac{\pi x}{2}\right) + \frac{4}{\pi^2}\cos\left(\frac{\pi x}{2}\right)\right]_0^1$$

$$V = 2\pi\left(\left[\frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{4}{\pi^2}\cos\left(\frac{\pi}{2}\right)\right] - \left[0 + \frac{4}{\pi^2}\cos(0)\right]\right)$$

$$V = 2\pi\left(\frac{2}{\pi} - \frac{4}{\pi^2}\right)$$

$$V = \boxed{4 - \frac{8}{\pi}}$$