8.2 Integrating Powers of Trig. Functions

1. $\int \cos^m x \sin^n x \, dx \quad (m, n \text{ positive integers})$

2. $\int \tan^m x \sec^n x \, dx \quad (m, n \text{ positive integers})$

3. $\int \sin(mx)\sin(nx)\,dx \qquad \int \cos(mx)\cos(nx)\,dx \qquad \int \sin(mx)\cos(nx)\,dx$

$(m, n \text{ integers with } m \neq n)$

8.2 Integrating Powers of Trig. Functions

1. $\int \cos^m x \sin^n x \, dx \quad (m, n \text{ positive integers})$

A) m, n: **one odd / one even** $\qquad ex: \int \cos^5 x \sin^2 x dx$

1. factor out one power from the trig. function that has the odd power

$$ex: \cos^5 x = \cos^4 x \cdot \cos x$$

2. use $\cos^2 x + \sin^2 x = 1$ to transform the remaining even power of the above trig function into the other trig. function

$$ex: \cos^5 x = (\cos^2 x)^2 \cos x = (1 - \sin^2 x)^2 \cos x$$

3. use u-substitution to finish the problem (let $u =$ "other" trig function)

$$\int \cos^5 x \sin^2 x dx = \int \sin^2 x \underbrace{(1 - \sin^2 x)^2 \cos x}_{\cos^5 x} dx \qquad u = \sin x \quad du = \cos x dx$$

$$= \int u^2 (1 - u^2)^2 \, du = \int u^2 (1 - 2u^2 + u^4) du = \int (u^2 - 2u^4 + u^6) du$$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C = \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$$

$\int \cos^m x \sin^n x\, dx \quad (m, n \text{ positive integers})$

B) m,n: **both odd** $\qquad ex: \int \cos^3 x \sin^3 x dx$

1. choose one of the trig. functions and factor out one power

$$ex: \sin^3 x = \sin^2 x \cdot \sin x$$

2. use $\cos^2 x + \sin^2 x = 1$ to transform the remaining even power of the above trig function into the other trig. function

$$ex: \sin^3 x = \sin^2 x \cdot \sin x = (1 - \cos^2 x)\sin x$$

3. use u – substitution to finish the problem (let u = "other" trig function)

$$\int \cos^3 x \sin^3 x dx = \int \cos^3 x \underbrace{(1-\cos^2 x)\sin x}_{\sin^3 x} dx \qquad u = \cos x, \quad du = -\sin x dx$$

$$= -\int u^3 (1-u^2) du = \int u^3 (u^2 - 1) du = \int (u^5 - u^3) du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C = \boxed{\tfrac{1}{6}\cos^6 x - \tfrac{1}{4}\cos^4 x + C}$$

$\int \cos^m x \sin^n x\, dx \quad (m, n \text{ positive integers})$

C) m,n: **both even** $\qquad ex: \int_0^\pi \cos^4 x \sin^2 x dx$

1. replace all even powers using the half-angle identities:
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$ex: \int_0^\pi \cos^4 x \sin^2 x dx = \int_0^\pi \cos^2 x \cos^2 x \sin^2 x dx$$

$$= \int_0^\pi \tfrac{1}{2}(1+\cos 2x)\tfrac{1}{2}(1+\cos 2x)\tfrac{1}{2}(1-\cos 2x) dx$$

$$= \tfrac{1}{8}\int_0^\pi (1+\cos 2x)(1+\cos 2x)(1-\cos 2x) dx$$

$$= \tfrac{1}{8}\int_0^\pi (1+\cos 2x - \cos^2 2x - \cos^3 2x) dx$$

$$= \tfrac{1}{8}\Bigg[\underbrace{\int_0^\pi dx}_{A} + \underbrace{\int_0^\pi \cos 2x dx}_{B} - \underbrace{\int_0^\pi \cos^2 2x dx}_{C} - \underbrace{\int_0^\pi \cos^3 2x dx}_{D}\Bigg]$$

think of $\cos 2x$ as w:

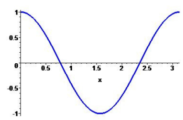
$$(1+w)(1+w)(1-w)$$
$$= (1+w)(1-w^2)$$
$$= 1 - w^2 + w - w^3$$

$ex: \int_0^{\pi} \cos^4 x \sin^2 x dx \; = \frac{1}{8}\left[A + B - C - D\right]$

$A = \int_0^{\pi} dx \qquad B = \int_0^{\pi} \cos 2x dx \qquad C = \int_0^{\pi} \cos^2 2x dx \qquad D = \int_0^{\pi} \cos^3 2x dx$

$\boxed{A = \pi}$

$\boxed{B = 0}$

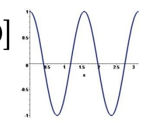


$C = \frac{1}{2}\int_0^{\pi} (1 + \cos 4x) dx$

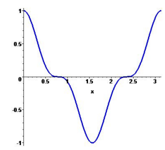
$= \frac{1}{2}\left[\int_0^{\pi} dx + \int_0^{\pi} \cos 4x dx\right]$

$= \frac{1}{2}\left[\pi + 0\right]$

$\boxed{C = \frac{\pi}{2}}$



$\boxed{D = 0}$



$\int_0^{\pi} \cos^4 x \sin^2 x dx = \frac{1}{8}\left[\pi - \frac{\pi}{2}\right] \; = \boxed{\frac{\pi}{16}}$

$\int \cos^m x \sin^n x \, dx \quad (m, n$ positive integers$)$

D) m, n : **one or both = 1** $\qquad ex: \int_0^{\pi} \cos^{10} x \sin x dx$

1. Just use $u-$substitution (let $u =$ the trig function with power $\neq 1$)
   (if both $= 1$, choose either)

$ex: \int_0^{\pi} \cos^{10} x \sin x dx$

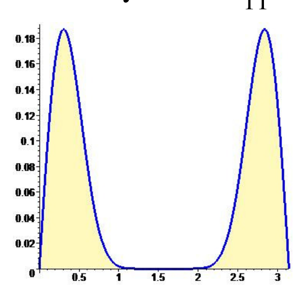
$u = \cos x$

$du = -\sin x dx$

$= -\int u^{10} du = -\frac{u^{11}}{11}$

$= \frac{-1}{11}\left[\cos^{11} x\right]_0^{\pi} \quad = \frac{-1}{11}\left[(-1)^{11} - 1^{11}\right]$

$= \frac{-1}{11}\left[-1 - 1\right] \; = \boxed{\frac{2}{11}}$



2. $\int \tan^m x \sec^n x\, dx \quad (m, n \text{ positive integers})$

A) m (the power of $\tan x$): **odd** $\qquad ex: \int \tan^3 x \sec^3 x dx$

1. factor out one power of $\sec x$ and one power of $\tan x$

$ex: \tan^3 x \sec^3 x = \tan^2 x \sec^2 x \sec x \tan x$

2. use $\tan^2 x = \sec^2 x - 1$ to transform the remaining even power of $\tan x$ to be in terms of $\sec x$

$ex: \tan^2 x \sec^2 x \sec x \tan x = (\sec^2 x - 1)\sec^2 x \sec x \tan x$

3. use u-substitution to finish the problem (let $u = \sec x$)

$ex: \int \tan^3 x \sec^3 x dx = \int (\sec^2 x - 1)\sec^2 x \sec x \tan x\, dx \qquad u = \sec x$

$du = \sec x \tan x dx$

$= \int u^2(u^2-1)du = \int (u^4 - u^2)du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

$\int \tan^m x \sec^n x\, dx \quad (m, n \text{ positive integers})$

B) n (the power of $\sec x$): **even** $\qquad ex: \int \tan^2 x \sec^4 x dx$

1. factor out $\sec^2 x$

$ex: \tan^2 x \sec^4 x = \tan^2 x \sec^2 x \sec^2 x$

2. use $\sec^2 x = 1 + \tan^2 x$ to transform the remaining even power of $\sec x$ to be in terms of $\tan x$

$ex: \tan^2 x \sec^2 x \sec^2 x = \tan^2 x (1 + \tan^2 x)\sec^2 x$

3. use u-substitution to finish the problem (let $u = \tan x$)

$ex: \int \tan^2 x \sec^4 x dx = \int \tan^2 x (1 + \tan^2 x)\sec^2 x\, dx \qquad u = \tan x$

$du = \sec^2 x dx$

$= \int u^2(1+u^2)du = \int (u^2 + u^4)du$

$= \frac{u^3}{3} + \frac{u^5}{5} + C = \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$

$$\int \tan^m x \sec^n x \, dx$$

C) For all other cases, there is no set method ☹

Here are some examples:

$$ex: \int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$= -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C$$

$$\boxed{\int \tan x dx = \ln|\sec x| + C}$$

$u = \cos x$

$du = -\sin x dx$

$$\int \frac{-1}{u} du$$

$$= -\ln|u| + C$$

$$ex: \int \sec x dx = \int \sec x \cdot \underbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}_{\text{convenient version of 1}} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$u = \sec x + \tan x$

$du = \left(\sec x \tan x + \sec^2 x\right) dx$

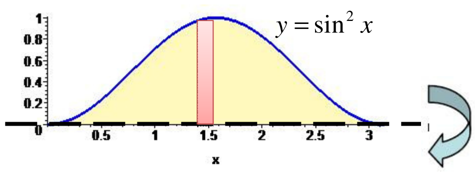
$$\int \frac{1}{u} du = \ln|u| + C$$

$$\boxed{\int \sec x dx = \ln|\sec x + \tan x| + C}$$

Also look at examples 7 and 8 from the text.

Math 104 – Rimmer
8.2 Integrating Powers of Trig. Functions

Find the volume of the solid obtained by rotating the region bounded by $y = \sin^2 x$ and $y = 0$ for $0 \le x \le \pi$ about the x-axis.



no gap b/w axis of rotation and the region

$\Rightarrow$ **Disk Method**

Radius: $r(x) = \sin^2 x$

$$Volume = \pi \int_a^b \left[r(x)\right]^2 dx$$

$$Volume = \pi \int_0^\pi \left[\sin^2 x\right]^2 dx$$

$$= \pi \int_0^\pi \sin^2 x \cdot \sin^2 x \, dx = \pi \int_0^\pi \tfrac{1}{2}(1-\cos 2x) \cdot \tfrac{1}{2}(1-\cos 2x) dx$$

$$= \tfrac{\pi}{4} \int_0^\pi (1-\cos 2x)(1-\cos 2x) dx = \tfrac{\pi}{4} \int_0^\pi \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \tfrac{\pi}{4} \int_0^\pi \left(1 - 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)\right) dx = \tfrac{\pi}{4} \int_0^\pi \left(\tfrac{3}{2} - \underbrace{2\cos 2x}_{0} + \underbrace{\tfrac{1}{2}\cos 4x}_{0}\right) dx$$

$$= \tfrac{\pi}{4} \int_0^\pi \tfrac{3}{2} dx = \boxed{\frac{3\pi^2}{8}}$$

m, n integers with $m \neq n$

3. $\int \sin(mx)\sin(nx)\,dx \qquad \int \cos(mx)\cos(nx)\,dx \qquad \int \sin(mx)\cos(nx)\,dx$

We change the product into a sum using the following identities:

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos([m-n]x) - \cos([m+n]x)\right]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos([m-n]x) + \cos([m+n]x)\right]$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin([m-n]x) + \sin([m+n]x)\right]$$

$$\int \sin(3x)\cos(5x)\,dx = \frac{1}{2}\int\left[\sin([3-5]x) + \sin([3+5]x)\right]dx$$

$$= \frac{1}{2}\int\left[\sin(-2x) + \sin(8x)\right]dx = \frac{1}{2}\int\left[\sin(8x) - \sin(2x)\right]dx$$

$$= \frac{1}{2}\left[-\tfrac{1}{8}\cos(8x) + \tfrac{1}{2}\cos(2x)\right] + C = \boxed{-\tfrac{1}{16}\cos(8x) + \tfrac{1}{4}\cos(2x) + C}$$