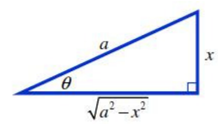
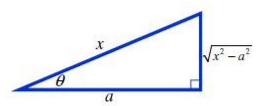
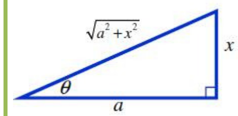
8.3 Trig. Substitution

For Integrals Involving	Substitution	Reference Triangle
$\sqrt{a^2-x^2}$ **Identity** $1-\sin^2\theta=\cos^2\theta$	$x=a\sin\theta \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ $\sqrt{a^2-x^2}=a\cos\theta$ since $x^2=a^2\sin^2\theta, \sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}$ $=\sqrt{a^2(1-\sin^2\theta)}=\sqrt{a^2\cos^2\theta}=a\cos\theta$	
$\sqrt{x^2-a^2}$ **Identity** $\sec^2\theta-1=\tan^2\theta$	$x=a\sec\theta \quad 0\le\theta\le\frac{\pi}{2}$, or $\pi\le\theta\le\frac{3\pi}{2}$ $\sqrt{x^2-a^2}=a\tan\theta$ since $x^2=a^2\sec^2\theta, \sqrt{x^2-a^2}=\sqrt{a^2\sec^2\theta-a^2}$ $=\sqrt{a^2(\sec^2\theta-1)}=\sqrt{a^2\tan^2\theta}=a\tan\theta$	
$\sqrt{a^2+x^2}$ **Identity** $1+\tan^2\theta=\sec^2\theta$	$x=a\tan\theta \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ $\sqrt{a^2+x^2}=a\sec\theta$ since $x^2=a^2\tan^2\theta, \sqrt{a^2+x^2}=\sqrt{a^2+a^2\tan^2\theta}$ $=\sqrt{a^2(1+\tan^2\theta)}=\sqrt{a^2\sec^2\theta}=a\sec\theta$	

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} \qquad \int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$$

$x=2\sin\theta$

$dx=2\cos\theta\, d\theta \qquad \sqrt{4-x^2}=2\cos\theta$

$x^2=4\sin^2\theta \qquad \sqrt{4-4\sin^2\theta}=\sqrt{4(1-\sin^2\theta)}=\sqrt{4\cos^2\theta}=2\cos\theta$

Limit Switching

$x=\sqrt{2}\Rightarrow\sqrt{2}=2\sin\theta \quad \sin\theta=\frac{\sqrt{2}}{2}\Rightarrow\theta=\frac{\pi}{4}$

$x=1\Rightarrow 1=2\sin\theta \quad \sin\theta=\frac{1}{2} \quad \Rightarrow\theta=\frac{\pi}{6}$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cancel{2\cos\theta}\,d\theta}{4\sin^2\theta\cdot\cancel{2\cos\theta}}$$

$$=\frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\csc^2\theta\, d\theta=-\frac{1}{4}\left[\cot\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$=-\frac{1}{4}\left[\frac{\cos\theta}{\sin\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}=-\frac{1}{4}\left(1-\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$$

$$=\boxed{\frac{1}{4}\left(\sqrt{3}-1\right)}$$

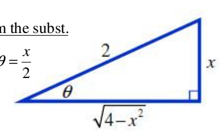
Using Triangle

$$=\int\frac{\cancel{2\cos\theta}\,d\theta}{4\sin^2\theta\cdot\cancel{2\cos\theta}} \qquad =\frac{1}{4}\int\csc^2\theta\, d\theta$$

$$=-\frac{1}{4}\left[\cot\theta\right]$$

from the subst.

$\sin\theta=\frac{x}{2}$



$\cot\theta=\frac{adj.}{opp.}$

$$=-\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}}=-\frac{1}{4}\left[\frac{\sqrt{2}}{\sqrt{2}}-\frac{\sqrt{3}}{1}\right]$$

$$=-\frac{1}{4}\left[1-\sqrt{3}\right]=\boxed{\frac{1}{4}\left(\sqrt{3}-1\right)}$$

$$\int \frac{5dx}{\sqrt{25x^2-9}} = \int \frac{5dx}{\sqrt{25\left(x^2-\frac{9}{25}\right)}} = \int \frac{\cancel{5}dx}{\cancel{5}\sqrt{x^2-\frac{9}{25}}} = \int \frac{dx}{\sqrt{x^2-\frac{9}{25}}}$$

$$= \int \frac{dx}{\sqrt{x^2-\frac{9}{25}}} \qquad x = \frac{3}{5}\sec\theta \qquad \sqrt{x^2-\frac{9}{25}} = \frac{3}{5}\tan\theta$$

$$dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta$$

$$\sqrt{\tfrac{9}{25}\sec^2\theta-\tfrac{9}{25}} = \sqrt{\tfrac{9}{25}\left(\sec^2\theta-1\right)} = \sqrt{\tfrac{9}{25}\tan^2\theta} = \frac{3}{5}\tan\theta$$

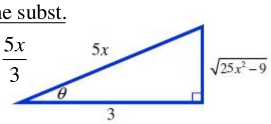
$$= \int \frac{\cancel{\tfrac{3}{5}}\sec\theta\cancel{\tan\theta}\, d\theta}{\cancel{\tfrac{3}{5}}\cancel{\tan\theta}} = \int \sec\theta\, d\theta$$

$$\boxed{\int \sec x\, dx = \ln\left|\sec x + \tan x\right| + C}$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

from the subst.

$$\sec\theta = \frac{5x}{3}$$



$$= \boxed{\ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C}$$

Complete the square

$$\int_1^2 \frac{dx}{\sqrt{4x-x^2}}$$

$$4x - x^2 = -x^2 + 4x \qquad\qquad 4x - x^2 = 4 - (x-2)^2$$

$$= -\left(x^2 - 4x + \underline{4}\right) + \underline{4}$$

$$= -(x-2)^2 + \underline{4}$$

$$= \int_1^2 \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$u = x - 2 \qquad x = 2 \Rightarrow u = 0$$

$$du = dx \qquad x = 1 \Rightarrow u = -1$$

$$= \int_{-1}^{0} \frac{du}{\sqrt{4-u^2}} \qquad \int_{-1}^{0} \frac{du}{\sqrt{4-u^2}} \qquad u = 2\sin\theta \qquad \sqrt{4-u^2} = 2\cos\theta$$

$$du = 2\cos\theta\, d\theta \qquad \sqrt{4-4\sin^2\theta} = \sqrt{4\left(1-\sin^2\theta\right)} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

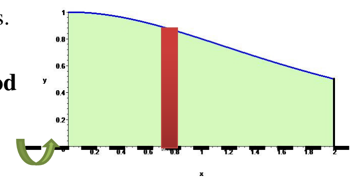
$$= \int_{-\frac{\pi}{6}}^{0} \frac{\cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}}$$

$$u = 0 \Rightarrow 0 = 2\sin\theta \qquad \sin\theta = 0 \qquad \Rightarrow \theta = 0$$

$$u = -1 \Rightarrow -1 = 2\sin\theta \qquad \sin\theta = -\frac{1}{2} \qquad \Rightarrow \theta = -\frac{\pi}{6}$$

$$= \int_{-\frac{\pi}{6}}^{0} d\theta = \left[\theta\right]_{-\frac{\pi}{6}}^{0} = \left(0 - \left(-\frac{\pi}{6}\right)\right) = \boxed{\frac{\pi}{6}}$$

Find the volume of the solid generated by revolving the region bounded by the curves $y=\dfrac{4}{x^2+4}, y=0, x=0,$ and $x=2$ about the $x-$axis.



no gap b/w axis of rotation and the region $\Rightarrow$ **Disk Method**

$$\text{Radius: } r(x)=\frac{4}{x^2+4}$$

$$Volume=\pi\int_a^b\left[r(x)\right]^2dx$$

$$Volume=\pi\int_0^2\left[\frac{4}{x^2+4}\right]^2dx=16\pi\int_0^2\frac{dx}{\left(x^2+4\right)^2}$$

$$=\cancel{16}\pi\int_0^{\frac{\pi}{4}}\frac{\cancel{2\sec^2\theta}d\theta}{\cancel{16}\cancel{\sec^4\theta}}=2\pi\int_0^{\frac{\pi}{4}}\cos^2\theta d\theta$$

$\sec^2\theta$

$$=2\pi\int_0^{\frac{\pi}{4}}\frac{1}{2}(1+\cos 2\theta)\,d\theta=\pi\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)\,d\theta$$

$$=\pi\left[\theta+\tfrac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{4}}=\pi\left[\left(\frac{\pi}{4}+\tfrac{1}{2}\sin\left(\frac{\pi}{2}\right)\right)-0\right]=\boxed{\pi\left(\frac{\pi}{4}+\frac{1}{2}\right)}$$

$$16\pi\int_0^2\frac{dx}{\left(x^2+4\right)^2}$$

$x=2\tan\theta$

$dx=2\sec^2\theta d\theta$

$x^2+4=4\tan^2\theta+4=4\left(1+\tan^2\theta\right)=4\sec^2\theta$

$\left(x^2+4\right)^2=16\sec^4\theta$

$x=2\Rightarrow 2=2\tan\theta$ $\quad$ $x=0\Rightarrow 0=2\tan\theta$

$\tan\theta=1$ $\quad$ $\tan\theta=0$

$\Rightarrow\theta=\dfrac{\pi}{4}$ $\quad$ $\Rightarrow\theta=0$