8.4 Partial Fraction Decomposition

Rational Function : $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials

The **degree** of a polynomial is the highest exponent on x

$$\left.\begin{matrix} x-4 \\ 2x+3 \end{matrix}\right\}\text{linear polynomials} \qquad \left.\begin{matrix} 2x^2-5x-12 \\ x^2-x+3 \end{matrix}\right\}\text{quadratic polynomials}$$

$2x^2-5x-12$ is called a **reducible** quadratic polynomial since it can be factored (over the reals)

$$2x^2-5x-12=(2x+3)(x-4) \qquad \text{the roots are real numbers, } b^2-4ac\ge 0$$

x^2-x+3 is called a **irreducible** quadratic polynomial since it cannot be factored (over the reals)

the roots are complex numbers, $b^2-4ac<0$

Every polynomial of degree $n>0$ with real coefficients can be written
as a product of linear and/or irreducible quadratic factors.

Goal: To integrate rational functions

- Write $q(x)$ in this manner and then express the rational function as the sum of simpler fractions.
- The simpler fractions should be easily integrable.

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simpler fractions : $\dfrac{1}{x-4} \qquad \dfrac{1}{(x-4)^2} \qquad \dfrac{1}{x^2+4}$

that can be integrated :

$$\int\frac{1}{x-4}\,dx = \ln|x-4|+C$$

$$\begin{matrix} u=x-4 \\ du=dx \end{matrix}\ \int\frac{1}{u}\,du=\ln|u|+C$$

$$\int\frac{1}{(x-4)^2}\,dx = \frac{-1}{x-4}+C$$

$$\begin{matrix} u=x-4 \\ du=dx \end{matrix}\ \int\frac{1}{u^2}\,du=\int u^{-2}\,du=\frac{-1}{u}+C$$

$$\int\frac{1}{x^2+4}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$\frac{1}{x^2+4}=\frac{1}{4\left(\frac{x^2}{4}+1\right)}=\frac{1}{4}\cdot\frac{1}{\left(\frac{x}{2}\right)^2+1}$$

$$\int\frac{1}{x^2+4}\,dx=\frac{1}{4}\int\frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \qquad \begin{matrix} u=\frac{x}{2} \\ du=\frac{1}{2}\,dx \end{matrix} \Rightarrow 2du=dx$$

$$=\frac{1}{4}\int\frac{2}{u^2+1}\,du=\frac{1}{2}\arctan u+C$$

Partial Fraction Decomposition: $\dfrac{p(x)}{q(x)}$

1. The degree of the denominator **must** be greater than the degree of the numerator
   If it is not, then **long divide** the denominator into the numerator.

2. Decompose the fraction in the following manner: $(A, B, C,$ and D are constants$)$

   $i)$ $q(x)$ can be written as a product of only linear polynomials

$$\frac{5x}{(x-4)(2x+3)}=\frac{A}{x-4}+\frac{B}{2x+3}$$

   $ii)$ $q(x)$ can be written as a product involving powers of linear polynomials

$$\frac{x^2+6x-4}{(x-3)^3(x+5)}=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}+\frac{D}{x+5}$$

   $iii)$ $q(x)$ can be written as a product involving irreducible quadratic polynomials

$$\frac{16x-5}{\left(x^2+2x+10\right)(x-7)}=\frac{Ax+B}{x^2+2x+10}+\frac{C}{x-7}$$

3. Use the method of **undetermined coefficients** to find the constants $A, B, C,$ and D

$$\int_3^4 \frac{4}{x^2-4}\,dx=\int_3^4 \frac{4}{(x+2)(x-2)}\,dx=\int_3^4\left(\frac{-1}{x+2}+\frac{1}{x-2}\right)dx$$

$$\frac{4}{(x+2)(x-2)}=\frac{A}{x+2}+\frac{B}{x-2}$$

$$x=-2\Rightarrow\frac{4}{\cancel{(x+2)}(-2-2)}\Rightarrow\boxed{A=-1}$$

$$x=2\Rightarrow\frac{4}{(2+2)\cancel{(x-2)}}\Rightarrow\boxed{B=1}$$

$$=\Big[-\ln|x+2|+\ln|x-2|\Big]_3^4$$

$$=\left[\ln\left|\frac{x-2}{x+2}\right|\right]_3^4$$

$$=\left[\ln\left|\frac{4-2}{4+2}\right|-\ln\left|\frac{3-2}{3+2}\right|\right]$$

$$=\ln\frac{2}{6}-\ln\frac{1}{5}$$

$$=\ln\left(\frac{\frac{1}{3}}{\frac{1}{5}}\right)=\boxed{\ln\left(\frac{5}{3}\right)}$$

$$\int \frac{x+8}{x(x+2)^2}\,dx$$

$$\frac{x+8}{x(x+2)^2}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}=\frac{A(x+2)^2}{x(x+2)^2}+\frac{Bx(x+2)}{(x+2)x(x+2)}+\frac{Cx}{x(x+2)^2}$$

$$\frac{x+8}{x(x+2)^2}=\frac{A(x+2)^2+Bx(x+2)+Cx}{x(x+2)^2}$$

Focus only on the numerators:

$$x+8=A(x+2)^2+Bx(x+2)+Cx$$

This equation should be true for all x. We cleverly choose values of x to plug in:

$$x=0: \quad 8=4A \Rightarrow \boxed{A=2}$$

$$x=-2: \quad 6=-2C \Rightarrow \boxed{C=-3}$$

$$x=1: \quad 9=9A+3B+C \Rightarrow 9=9(2)+3B+(-3) \Rightarrow 9=18+3B-3$$

$$\Rightarrow 9-15=3B \Rightarrow \boxed{B=-2}$$

$\int \frac{x+8}{x(x+2)^2}\,dx$ continued:

$$\frac{x+8}{x(x+2)^2}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \quad \Rightarrow \frac{x+8}{x(x+2)^2}=\frac{2}{x}-\frac{2}{x+2}-\frac{3}{(x+2)^2}$$

$$\int \frac{x+8}{x(x+2)^2}\,dx=\int\left(\frac{2}{x}-\frac{2}{x+2}-\frac{3}{(x+2)^2}\right)dx$$

$$=\boxed{2\ln|x|-2\ln|x+2|+\frac{3}{x+2}+C}$$

$$\int_2^3 \frac{x^3-x^2-1}{x^2-x}dx = \int_2^3\left[x+\frac{-1}{x(x-1)}\right]dx = \int_2^3\left[x+\frac{1}{x}+\frac{-1}{x-1}\right]dx$$

$$\begin{array}{r} x \\ x^2-x\overline{)\;x^3-x^2+0x-1} \\ \underline{-\left(x^3-x^2\right)} \quad\quad \\ \downarrow\; -1 \end{array}$$

$$\Rightarrow \frac{x^3-x^2-1}{x^2-x} = x+\frac{-1}{x^2-x}$$

$$\frac{-1}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1} = \frac{1}{x}+\frac{-1}{x-1}$$

$$x=0 \Rightarrow A = \frac{-1}{\cancel{x}(0-1)} = 1$$

$$x=1 \Rightarrow B = \frac{-1}{1\cancel{(x-1)}} = -1$$

$$= \left[\frac{x^2}{2}+\ln|x|-\ln|x-1|\right]_2^3$$

$$= \left[\frac{x^2}{2}+\ln\left(\frac{x}{x-1}\right)\right]_2^3$$

$$= \left[\frac{9}{2}+\ln\left(\frac{3}{2}\right)\right]-\left[\frac{4}{2}+\ln(2)\right]$$

$$= \frac{5}{2}+\ln\left(\frac{3}{2}\right)-\ln 2$$

$$= \frac{5}{2}+\ln\left(\frac{3/2}{2}\right)$$

$$= \boxed{\frac{5}{2}+\ln\left(\tfrac{3}{4}\right)}$$

$$\int \frac{3-x}{x\left(x^2+1\right)}dx$$

$$\frac{3-x}{x\left(x^2+1\right)} = \frac{A}{x}+\frac{Bx+C}{x^2+1} = \frac{3}{x}+\frac{-3x-1}{x^2+1}$$

$$\frac{3-x}{x\left(x^2+1\right)} = \frac{A\left(x^2+1\right)}{x\left(x^2+1\right)}+\frac{(Bx+C)x}{x\left(x^2+1\right)}$$

$$3-x = A\left(x^2+1\right)+(Bx+C)x$$

$$3-x = Ax^2+A+Bx^2+Cx$$

$$3-x = (A+B)x^2+Cx+A$$

Match up coefficients from the left and right hand sides

$$0x^2-1x+3 = (A+B)x^2+Cx+A$$

$\Rightarrow A=3$

$\Rightarrow C=-1$

$\Rightarrow A+B=0 \quad \Rightarrow B=-3$

$$\int \frac{3-x}{x\left(x^2+1\right)}dx = \int\left(\frac{3}{x}+\frac{-3x-1}{x^2+1}\right)dx$$

$$= \int\left(\frac{3}{x}+\underbrace{\frac{-3x}{x^2+1}}_{\substack{u=x^2+1 \\ du=2xdx}\Rightarrow \frac{1}{2}du=xdx}+\frac{-1}{x^2+1}\right)dx$$

$$-\frac{3}{2}\int\frac{1}{u}du = -\frac{3}{2}\ln|u|+C$$

$$= \boxed{3\ln|x|-\frac{3}{2}\ln\left|x^2+1\right|-\arctan x+C}$$