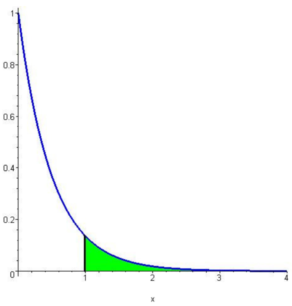
8.8 Improper Integrals

Infinite Upper Limit

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$\int_1^\infty e^{-2x}dx = \lim_{b\to\infty}\int_1^b e^{-2x}dx = \lim_{b\to\infty}\left[\frac{-1}{2}e^{-2x}\right]_1^b = \lim_{b\to\infty}\left[\frac{-1}{2e^{2x}}\right]_1^b$$



$$= \lim_{b\to\infty}\frac{-1}{2e^{2b}} + \frac{1}{2e^2}$$

$$= \boxed{\frac{1}{2e^2}} \quad \text{since } \lim_{b\to\infty}\frac{-1}{2e^{2b}} = 0$$

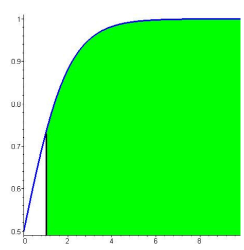
Infinite Upper Limit

$$\int_1^\infty \frac{e^x}{1+e^x}dx = \lim_{b\to\infty}\int_1^b \frac{e^x}{1+e^x}dx \qquad \boxed{\begin{array}{l} u = 1+e^x \\ du = e^x dx \end{array}} \qquad \boxed{\int \tfrac{1}{u}du = \ln|u| + C}$$

$$= \lim_{b\to\infty}\left[\ln\left(1+e^x\right)\right]_1^b$$

$$= \lim_{b\to\infty}\ln\left(1+e^b\right) - \ln\left(1+e\right) = \boxed{\infty}$$

DIVERGENT



Math 104 – Rimmer
8.8 Improper Integrals

Infinite Lower Limit

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx$$

D		I
x	+	e^x
1	-	e^x
0		e^x

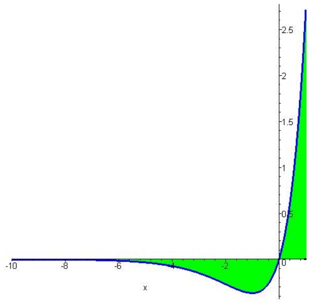
$$\int_{-\infty}^{1} xe^x\,dx = \lim_{a\to-\infty}\int_{a}^{1} xe^x\,dx = \lim_{a\to-\infty}\left[xe^x - e^x\right]_a^1$$

$$= \lim_{a\to-\infty}(e-e)-\left(ae^a - e^a\right)$$

$$= \lim_{a\to-\infty} e^a(1-a) = 0\cdot\infty \text{ (indeterminate)}$$

$$= \lim_{a\to-\infty}\frac{1-a}{e^{-a}} = \frac{\infty}{\infty}\ \text{(L'Hospital)}$$

$$\overset{L'H}{=} \lim_{a\to-\infty}\frac{-1}{-e^{-a}} = \boxed{0}$$



Math 104 – Rimmer
8.8 Improper Integrals

Infinite Upper and Lower Limit

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx \ ;c \text{ any real number}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{c} f(x)\,dx + \lim_{b\to\infty}\int_{c}^{b} f(x)\,dx$$

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^6}\,dx = \lim_{a\to-\infty}\int_{a}^{0}\frac{x^2}{1+x^6}\,dx + \lim_{b\to\infty}\int_{0}^{b}\frac{x^2}{1+x^6}\,dx$$

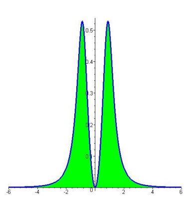
$u = x^3$
$du = 3x^2dx \quad \frac{1}{3}du = x^2dx$

$\frac{1}{3}\int\frac{1}{1+u^2}\,du = \frac{1}{3}\arctan u + C$

$$= \lim_{a\to-\infty}\left[\tfrac{1}{3}\arctan\left(x^3\right)\right]_a^0 + \lim_{b\to\infty}\left[\tfrac{1}{3}\arctan\left(x^3\right)\right]_0^b$$

$$= \lim_{a\to-\infty} -\tfrac{1}{3}\arctan\left(a^3\right) + \lim_{b\to\infty}\tfrac{1}{3}\arctan\left(b^3\right)$$

$$= -\tfrac{1}{3}\left(-\tfrac{\pi}{2}\right) + \tfrac{1}{3}\left(\tfrac{\pi}{2}\right) = \boxed{\frac{\pi}{3}}$$



Math 104 – Rimmer
8.8 Improper Integrals

Infinite Discontinuity at Lower Limit

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

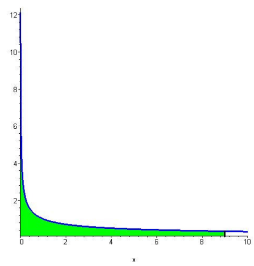
$f(a)\to$infinite discontinuity

$$\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^9 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^9 x^{-1/2}dx = \lim_{t\to 0^+}\left[2x^{1/2}\right]_t^9$$

$f(0)\to$infinite discontinuity

$$= \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^9$$

$$= \lim_{t\to 0^+} 6-2\sqrt{t} \quad = \boxed{6}$$



Math 104 – Rimmer
8.8 Improper Integrals

Infinite Discontinuity at Upper Limit

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

$f(b)\to$infinite discontinuity

$$\int_0^8 \frac{dx}{\sqrt[3]{8-x}} = \lim_{t\to 8^-}\int_0^t \frac{dx}{\sqrt[3]{8-x}} = \lim_{t\to 8^-}\int_0^t (8-x)^{-1/3}\,dx$$

$u = 8-x$
$du = -dx$

$-\int u^{-1/3}du = \frac{-3}{2}u^{2/3}+C$

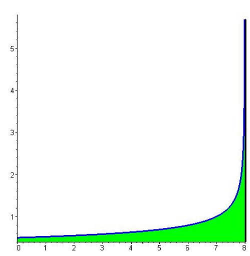
$f(8)\to$infinite discontinuity

$$= \lim_{t\to 8^-}\left[\tfrac{-3}{2}(8-x)^{2/3}\right]_0^t$$

$$= \lim_{t\to 8^-}\tfrac{-3}{2}(8-t)^{2/3}+\tfrac{3}{2}(8)^{2/3}$$

$$= \tfrac{3}{2}\left[(8)^{1/3}\right]^2 \quad \text{since } \lim_{t\to 8^-}\tfrac{-3}{2}(8-t)^{2/3}=0$$

$$= \boxed{6}$$



Infinite Discontinuity inside the interval

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$

$f(c)\to$infinite discontinuity $a<c<b$

$$\int_{-2}^{3}\frac{dx}{x^4} = \int_{-2}^{0}\frac{dx}{x^4} + \int_{0}^{3}\frac{dx}{x^4} = \lim_{t\to 0^-}\int_{-2}^{t}\frac{dx}{x^4} + \lim_{t\to 0^+}\int_{t}^{3}\frac{dx}{x^4}$$

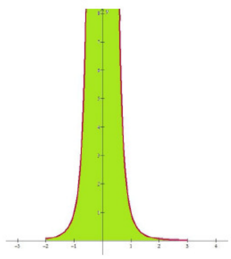
$f(0)\to$infinite discontinuity -2<0<3

$$= \lim_{t\to 0^-}\left[\frac{-1}{3x^3}\right]_{-2}^{t} + \lim_{t\to 0^+}\left[\frac{-1}{3x^3}\right]_{t}^{3}$$

$$= \boxed{\infty}$$

Both are **DIVERGENT**

(actually only need one of them to be divergent for the entire integral to be divergent)



Doubly Improper

$$\int_0^\infty f(x)\,dx = \int_0^c f(x)\,dx + \int_c^\infty f(x)\,dx = \lim_{a\to 0^+}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$$

$f(0)\to$infinite discontinuity

$$\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx = \int_0^1 \frac{e^{-1/x}}{x^2}\,dx + \int_1^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{e^{-1/x}}{x^2}\,dx + \lim_{b\to\infty}\int_1^b \frac{e^{-1/x}}{x^2}\,dx$$

$f(0)\to$infinite discontinuity

$u=\frac{-1}{x}$

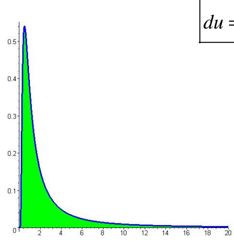
$du=\frac{1}{x^2}\,dx \quad \int e^u\,du = e^u + C$

$$= \lim_{a\to 0^+}\left[e^{-1/x}\right]_a^1 + \lim_{b\to\infty}\left[e^{-1/x}\right]_1^b$$

$$= \lim_{a\to 0^+}\left[\tfrac{1}{e^{1/x}}\right]_a^1 + \lim_{b\to\infty}\left[\tfrac{1}{e^{1/x}}\right]_1^b$$

$$= \left[\tfrac{1}{e} - \lim_{a\to 0^+}\left[\tfrac{1}{e^{1/a}}\right]\right] + \left[\lim_{b\to\infty}\left[\tfrac{1}{e^{1/b}}\right] - \tfrac{1}{e}\right]$$

$$= \frac{1}{e^{\lim_{b\to\infty}\frac{1}{b}}} - \frac{1}{e^{\lim_{a\to 0^+}\frac{1}{a}}} = 1 - 0 = \boxed{1}$$

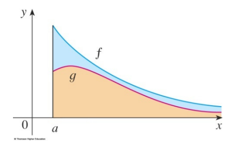


Comparison Theorem

Suppose that $f(x)$ and $g(x)$ are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

$a)$ If $\int_a^\infty f(x)\,dx$ is **convergent**, then $\int_a^\infty g(x)\,dx$ is **convergent**.

$b)$ If $\int_a^\infty g(x)\,dx$ is **divergent**, then $\int_a^\infty f(x)\,dx$ is **divergent**.



See problems 49-54 in section 8.8

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

50. $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

52. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

53. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

54. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

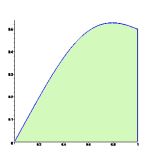
$$\frac{x}{x^3+1} \le \frac{x}{x^3} \text{ for } x \ge 1 \qquad \underbrace{\frac{x}{x^3+1}}_{g} \le \underbrace{\frac{1}{x^2}}_{f} \text{ for } x \ge 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b = \lim_{b\to\infty}\frac{-1}{b}+1 = \boxed{1} \textbf{ Convergent}$$

Thus $\displaystyle\int_1^\infty \frac{x}{x^3+1}\,dx$ is **convergent** by the Comparison Theorem.

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \underbrace{\int_0^1 \frac{x}{x^3+1}\,dx}_{\textbf{constant}} + \underbrace{\int_1^\infty \frac{x}{x^3+1}\,dx}_{\textbf{convergent}}$$

$$\boxed{\int_0^\infty \frac{x}{x^3+1}\,dx \text{ is } \textbf{convergent}.}$$



50. $\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

$\frac{2+e^{-x}}{x} \geq \frac{2}{x}$ for $x \geq 1$

$\underbrace{\frac{2+e^{-x}}{x}}_{f} \geq \underbrace{\frac{2}{x}}_{g}$ for $x \geq 1$

$\int_1^\infty \frac{2}{x}\,dx = \lim_{b\to\infty} 2\left[\ln x\right]_1^b = \lim_{b\to\infty} 2\left[\ln b\right] - 0 = \infty$ **Divergent**

Thus $\boxed{\int_1^\infty \frac{2+e^{-x}}{x}\,dx \text{ is } \textbf{divergent}}$ by the Comparison Theorem.