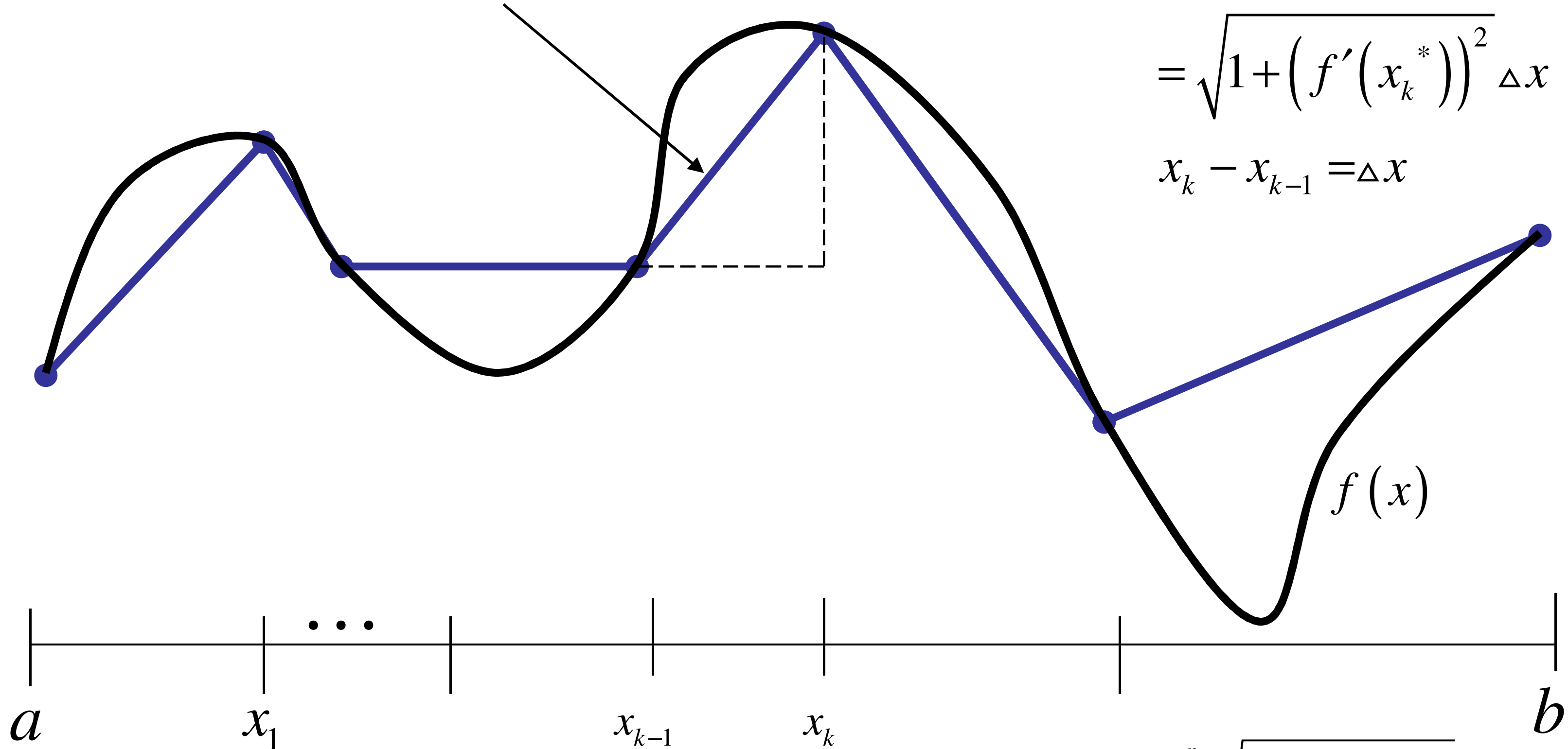
9.1 Arc Length

$$\sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} = \sqrt{(x_k - x_{k-1})^2 + (f'(x_k^*)(x_k - x_{k-1}))^2}$$

$$= \sqrt{1 + (f'(x_k^*))^2}\, \triangle x$$

$$x_k - x_{k-1} = \triangle x$$



by the Mean Value Theorem: $\exists\, x_k^* \in (x_{k-1}, x_k)$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

$$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1})$$

$$\text{Arc Length} = \sum_{k=1}^{n} \sqrt{1 + (f'(x_k^*))^2}\, \triangle x$$

$$\text{Arc Length} = \lim_{\triangle x \to 0} \sum_{k=1}^{n} \sqrt{1 + (f'(x_k^*))^2}\, \triangle x$$

$$\boxed{\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

Find the length of the curve.

$$y=\frac{2}{3}\left(1+x^2\right)^{3/2} \quad 0\le x\le 3$$

$$y'=2x\left(1+x^2\right)^{1/2}$$

$$\left(y'\right)^2=4x^2\left(1+x^2\right)$$

$$\sqrt{1+\left(y'\right)^2}=\sqrt{1+4x^2\left(1+x^2\right)}=\sqrt{4x^4+4x^2+1}=\sqrt{\left(2x^2+1\right)^2}=2x^2+1$$

$$\int_0^3\sqrt{1+\left(y'\right)^2}\,dx=\int_0^3\left(2x^2+1\right)dx=\left[\frac{2}{3}x^3+x\right]_0^3=\boxed{21}$$

Find the length of the curve $y = 1 + \frac{2}{3}(x-1)^{3/2}$ for $1 \le x \le 4$.

$$y = 1 + \frac{2}{3}(x-1)^{3/2}$$

$$\frac{dy}{dx} = (x-1)^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x-1$$

$$\sqrt{1+\left(\tfrac{dy}{dx}\right)^2} = \sqrt{1+x-1} = \sqrt{x}$$

$$\text{Arc Length} = \int_a^b \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx = \int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_1^4 = \frac{2}{3}\left[\left(4^{1/2}\right)^3 - 1\right] = \boxed{\frac{14}{3}}$$

Find the length of the curve.

$$y=\frac{1}{3}\sqrt{x}(3-x) \quad 0\le x\le 3 \qquad y=\sqrt{x}-\frac{1}{3}x^{3/2}$$

$$y'=\frac{1}{2\sqrt{x}}-\frac{1}{3}\cdot\frac{3}{2}\sqrt{x} \;=\frac{1}{2\sqrt{x}}-\frac{\sqrt{x}}{2} \Rightarrow y'=\frac{1}{2\sqrt{x}}-\frac{\sqrt{x}}{2}$$

$$(y')^2=\left(\frac{1}{2\sqrt{x}}-\frac{\sqrt{x}}{2}\right)\left(\frac{1}{2\sqrt{x}}-\frac{\sqrt{x}}{2}\right) =\frac{1}{4x}-\frac{1}{4}-\frac{1}{4}+\frac{x}{4} \Rightarrow (y')^2=\frac{1}{4x}-\frac{1}{2}+\frac{x}{4}$$

$$\sqrt{1+(y')^2}=\sqrt{1+\left(\frac{1}{4x}-\frac{1}{2}+\frac{x}{4}\right)}=\sqrt{\frac{1}{4x}+\frac{1}{2}+\frac{x}{4}}=\sqrt{\left(\frac{1}{2\sqrt{x}}+\frac{\sqrt{x}}{2}\right)\left(\frac{1}{2\sqrt{x}}+\frac{\sqrt{x}}{2}\right)}$$

$$\Rightarrow \sqrt{1+(y')^2}=\frac{1}{2\sqrt{x}}+\frac{\sqrt{x}}{2}$$

$$\int_0^3\sqrt{1+(y')^2}\,dx=\int_0^3\left(\frac{1}{2\sqrt{x}}+\frac{\sqrt{x}}{2}\right)dx=\int_0^3\left(\frac{x^{-1/2}}{2}+\frac{x^{1/2}}{2}\right)dx=\left[x^{1/2}+\frac{1}{3}x^{3/2}\right]_0^3=3^{1/2}+\frac{3^{3/2}}{3}$$

$$=\sqrt{3}+\frac{3\sqrt{3}}{3}=\boxed{2\sqrt{3}}$$