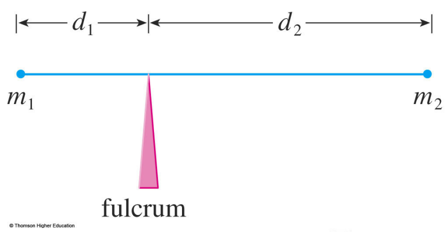
9.3 Center of Mass 1-d

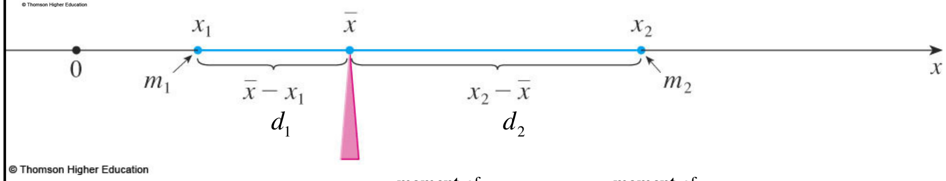


Archimedes'

Law of the Lever

the rod will balance if

$$\boxed{m_1 d_1 = m_2 d_2}$$



$$m_1\left(\bar{x} - x_1\right) = m_2\left(x_2 - \bar{x}\right)$$

$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2\bar{x}$$

$$\bar{x}\left(m_1 + m_2\right) = m_1 x_1 + m_2 x_2$$

moment of m_1 (with respect to the origin)

moment of m_2 (with respect to the origin)

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

moment of the system about the origin

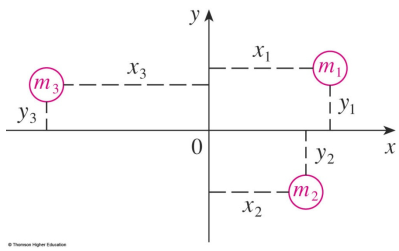
total mass

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\text{total mass}}}$$

$$\underbrace{\bar{x}\cdot\left(\text{total mass}\right)}_{\text{moment for the total mass}} = \underbrace{\sum_{i=1}^{n} m_i x_i}_{\text{moment for the system}}$$

If the total mass was concentrated at $\bar{x}$, then its moment would be the same as the moment for the system.

9.3 Center of Mass 2-d



M_y = moment of the system about the y-axis

measures the tendency of the system to rotate about the y-axis

$$M_y = m_1x_1 + m_2x_2 + m_3x_3$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \qquad \boxed{\bar{x} = \frac{M_y}{\text{total mass}}}$$

M_x = moment of the system about the x-axis

measures the tendency of the system to rotate about the x-axis

$$M_x = m_1y_1 + m_2y_2 + m_3y_3$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \qquad \boxed{\bar{y} = \frac{M_x}{\text{total mass}}}$$

$$M_y = \bar{x}(\text{total mass})$$

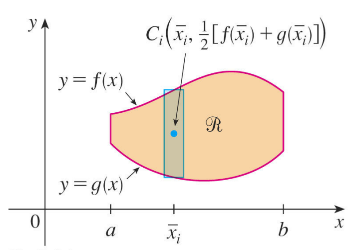
$$M_x = \bar{y}(\text{total mass})$$

The center of mass is the point $(\bar{x}, \bar{y})$ where a single particle with the same mass as the total mass would have the same moments as the system

9.3 Center of Mass 3-d

Consider a flat plate (called a lamina) with uniform density ρ that occupies a region $\Re$ of the plane.

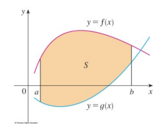
The center of mass of the plate is called the **centroid** of $\Re$.



General Formulas

Thin plate : region between $y = f(x)$ and $y = g(x)$ with $f(x) \ge g(x)$

Constant density function $\rho(x) = \rho$



Moment about the $x - axis$

$$M_x = \frac{\rho}{2}\int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

Moment about the $y - axis$

$$M_y = \rho\int_a^b x\cdot\left(f(x) - g(x)\right)dx$$

Mass

$$M = \rho\cdot\underbrace{\int_a^b \left(f(x) - g(x)\right)dx}_{\text{Area of the region b/w } f(x) \text{ and } g(x)}$$

Let $A = area$ of the region b/w $f(x)$ and $g(x)$

Center of Mass

$$\left(\bar{x}, \bar{y}\right)$$

$$\bar{x} = \frac{M_y}{M} = \frac{1}{A}\int_a^b x\cdot\left(f(x) - g(x)\right)dx$$

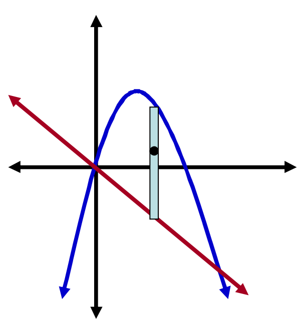
$$\bar{y} = \frac{M_x}{M} = \frac{1}{2A}\int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

Thin plate : region between $y = x - x^2$ and $y = -x$

Constant density function $\rho(x) = \rho$

Limits of integration: $x - x^2 = -x$

$$2x - x^2 = 0$$
$$x(2 - x) = 0$$
$$x = 0, \quad x = 2$$



Moment about the $x -$ axis

$$M_x = \frac{\rho}{2}\int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

$$= \frac{\rho}{2}\int_0^2 \left[\left(x - x^2\right)^2 - (-x)^2\right]dx$$

$$= \frac{\rho}{2}\int_0^2 \left[\left(\cancel{x^2} - 2x^3 + x^4\right) - \cancel{x^2}\right]dx$$

$$= \frac{\rho}{2}\int_0^2 \left(-2x^3 + x^4\right)dx = \frac{\rho}{2}\left(\frac{-x^4}{2} + \frac{x^5}{5}\right)_0^2 = \frac{\rho}{2}\left(-8 + \frac{32}{5}\right) = \frac{\rho}{2}\left(\frac{-40 + 32}{5}\right)$$

$$= \frac{\rho}{2}\left(\frac{-8}{5}\right) = \boxed{\frac{-4}{5}\rho}$$

Moment about the $y-axis$

$$\boxed{M_y = \rho\int_a^b x\cdot\left(f(x)-g(x)\right)dx}$$

$$= \rho\int_0^2 x\left[\left(x-x^2\right)-\left(-x\right)\right]dx = \rho\int_0^2 x\cdot\left(2x-x^2\right)dx = \rho\int_0^2\left(2x^2-x^3\right)dx$$

$$= \rho\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)_0^2 = \rho\left(\frac{16}{3}-\frac{16}{4}\right) = \rho\left(\frac{16}{3}-4\right) = \rho\left(\frac{16-12}{3}\right) = \boxed{\frac{4}{3}\rho}$$

Mass

$$\boxed{M = \rho\cdot\underbrace{\int_a^b\left(f(x)-g(x)\right)dx}_{\text{Area of the region b/w } f(x) \text{ and } g(x)}}$$

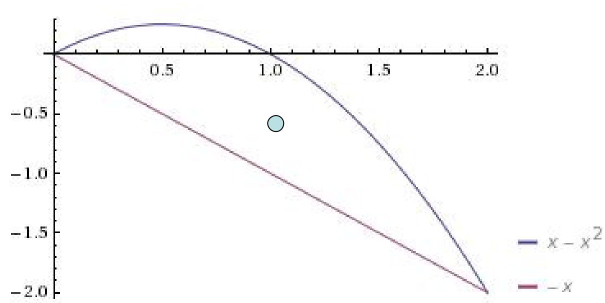
$$= \rho\cdot\int_0^2\left(2x-x^2\right)dx = \rho\left(x^2-\frac{x^3}{3}\right)_0^2 = \rho\left(4-\frac{8}{3}\right) = \rho\left(\frac{12-8}{3}\right) = \boxed{\frac{4}{3}\rho}$$

Math 104 – Rimmer
9.3 Center of Mass

Center of Mass

$$\bar{x} = \frac{M_y}{M} \quad \bar{y} = \frac{M_x}{M} \qquad \bar{x} = \frac{\frac{4}{3}\delta}{\frac{4}{3}\delta} \quad \bar{y} = \frac{\frac{-4}{5}\delta}{\frac{4}{3}\delta} \qquad \bar{x} = 1 \quad \bar{y} = \frac{-3}{5}$$

$$\left(\bar{x},\bar{y}\right) = \left(1,\frac{-3}{5}\right)$$

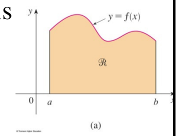


General Formulas

Thin plate : region under the graph of $y = f(x)$ and above the $x-$axis

Constant density function $\rho(x) = \rho$

Set $g(x) = 0$ in the previous formulas.



(a)

$$\boxed{M_x = \frac{\rho}{2}\int_a^b \left[f(x)\right]^2 dx}$$

$$\boxed{M_y = \rho\int_a^b x f(x)\,dx}$$

$$\boxed{M = \rho\underbrace{\int_a^b f(x)\,dx}_{\text{Area under } f(x)}}$$

Let $A = area$ under $f(x)$

Center of Mass

$$(\bar{x}, \bar{y})$$

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\,dx}$$

$$\boxed{\bar{y} = \frac{1}{2A}\int_a^b \left[f(x)\right]^2 dx}$$