9.5 Probability Density Function

A **random variable**, usually written *X*, is a variable whose possible values are numerical outcomes of a random phenomenon.

A **continuous random variable** is one which takes an infinite number of possible values (usually measurements)

A **probability density function** is a function *f* defined for all real *x* and having the following properties:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

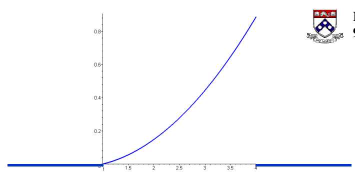
» Every continuous random variable, *X*, has a probability density function.

» Used to determine the probability that a continuous random variable lies between two values

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)\,dx}$$

Math 104 – Rimmer
9.5 Probability

$$f(x) = \begin{cases} \frac{2}{27}x(x-1) & 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



a) Verify that *f* is a probability density function.

$f(x) \geq 0$ for all x

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{27}\int_1^4 x(x-1)\,dx = \frac{2}{27}\int_1^4 (x^2 - x)\,dx = \frac{2}{27}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_1^4$$

$$= \frac{2}{27}\left(\left[\frac{64}{3} - 8\right] - \left[\frac{1}{3} - \frac{1}{2}\right]\right) = \frac{2}{27}\left(21 - 8 + \frac{1}{2}\right) = \frac{2}{27}\left(\frac{27}{2}\right) = \boxed{1}$$

b) Find $P(2 \leq x \leq 3)$.

$$P(2 \leq x \leq 3) = \frac{2}{27}\int_2^3 x(x-1)\,dx = \frac{2}{27}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_2^3$$

$$= \frac{2}{27}\left(\left[9 - \frac{9}{2}\right] - \left[\frac{8}{3} - 2\right]\right) = \frac{2}{27}\left(\frac{66 - 27 - 16}{6}\right)$$

$$= \boxed{\frac{23}{81} \approx 0.284}$$

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9.5 Probability

$$f(x)=\begin{cases} kx^2 & 2\le x\le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find k such that $f(x)$ is a probability density function.

$$f(x)\ge 0 \text{ for all } x \;\Rightarrow\; \boxed{k\ge 0}$$

$$\int_{-\infty}^{\infty} f(x)\,dx=1 \Rightarrow \int_{2}^{5} kx^2\,dx=1$$

$$\Rightarrow k\left[\frac{x^3}{3}\right]_2^5=1 \Rightarrow k\left[\frac{125-8}{3}\right]=1$$

$$\Rightarrow k\left[\frac{117}{3}\right]=1 \;\Rightarrow 39k=1 \;\Rightarrow \boxed{k=\frac{1}{39}}$$

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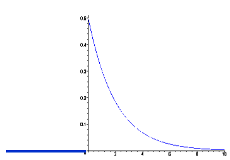
Exponential Density Function

$$f(x)=\begin{cases} 0 & x<0 \\ ke^{-kx} & x\ge 0 \end{cases},\; k>0$$



Examples of applications:

- » life span of electronic components
- » duration of telephone calls
- » waiting time in a doctor's office
- » time b/w successive flight arrivals and departures in an airport

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{0}^{\infty} ke^{-kx}\,dx \;=\lim_{b\to\infty}\int_{0}^{b} ke^{-kx}\,dx \;=\lim_{b\to\infty}\left(\cancel{k}\cdot\frac{-1}{\cancel{k}}e^{-kx}\right)_0^b$$

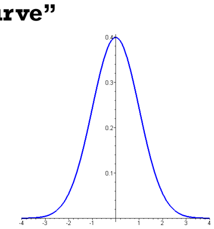
$$=\lim_{b\to\infty}\left(\frac{-1}{e^{kx}}\right)_0^b=\lim_{b\to\infty}\left(\underbrace{\frac{-1}{e^{kb}}}_{\to 0}-(-1)\right)=\boxed{1}$$

Normal Density Function **"bell curve"**

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

$\mu=$ mean

$\sigma=$ standard deviation



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Mean (average value) or expected value of a probability density function, a measure of the center of a pdf.

$$\boxed{\mu=\int_{-\infty}^{\infty} x f(x)\,dx}$$

$$f(x)=\begin{cases}\frac{2}{27}x(x-1) & 1\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

Find the mean.

$$\mu=\int_{-\infty}^{\infty} x f(x)\,dx=\frac{2}{27}\int_1^4 x^2(x-1)\,dx=\frac{2}{27}\left(\tfrac{x^4}{4}-\tfrac{x^3}{3}\right)\Big|_1^4$$

$$=\frac{2}{27}\left[\left(64-\tfrac{64}{3}\right)-\left(\tfrac{1}{4}-\tfrac{1}{3}\right)\right]=\frac{2}{27}\left[64-21-\tfrac{1}{4}\right]=\tfrac{2}{27}\left[\tfrac{171}{4}\right]=\boxed{\frac{19}{6}}$$

$$f(x)=\begin{cases}0 & x<0\\ ke^{-kx} & x\ge 0\end{cases},\quad k>0$$

Find the mean.

$$\mu=\int_{-\infty}^{\infty} x f(x)\,dx=\int_0^{\infty} kxe^{-kx}\,dx=\lim_{b\to\infty}\int_0^b kxe^{-kx}\,dx$$

$$=\lim_{b\to\infty}\left[-xe^{-kx}-\tfrac{1}{k}e^{-kx}\right]_0^b$$

$$=\lim_{b\to\infty}\left[\left(\frac{-b}{e^{kb}}\nearrow 0-\frac{1}{ke^{kb}}\nearrow 0\right)-\left(0-\frac{1}{k}\right)\right]$$

$$=\boxed{\frac{1}{k}}$$

$$\boxed{\begin{array}{ll} u=kx & dv=e^{-kx}\\ du=k\,dx & v=\frac{-1}{k}e^{-kx}\\ \multicolumn{2}{l}{=-xe^{-kx}+\int e^{-kx}dx}\\ \multicolumn{2}{l}{=-xe^{-kx}-\frac{1}{k}e^{-kx}}\end{array}}$$

$$\boxed{\begin{array}{l}\lim\limits_{b\to\infty}\dfrac{-b}{e^{kb}}=\dfrac{"\infty"}{\infty}\\ \overset{L'H}{=}\lim\limits_{b\to\infty}\dfrac{-1}{ke^{kb}}=0\end{array}}$$

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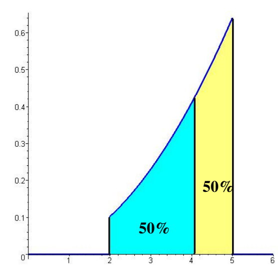
Median (m) of a probability density function is a number such that ½ the area under the graph of f lies to the right of it.

$$\boxed{\int_m^{\infty} f(x)\,dx=\frac{1}{2}}$$

$$f(x)=\begin{cases}\frac{1}{39}x^2 & 2\le x\le 5\\ 0 & \text{otherwise}\end{cases}$$

Find the median.



$$\int_m^{\infty}\frac{x^2}{39}dx=\frac{1}{2}\;\Rightarrow\int_m^{5}\frac{x^2}{39}dx=\frac{1}{2}\;\Rightarrow\left[\frac{x^3}{117}\right]_m^5=\frac{1}{2}\;\Rightarrow\frac{125-m^3}{117}=\frac{1}{2}$$

$$\Rightarrow 250-2m^3=117$$

$$\Rightarrow 2m^3=133$$

$$\Rightarrow\boxed{m=\sqrt[3]{\frac{133}{2}}\approx 4.05}$$

Exponential Density Function

$$f(x)=\begin{cases}0 & x<0\\ ke^{-kx} & x\geq 0\end{cases},\ k>0 \qquad \mu=\frac{1}{k}$$

Say μ is known $\Rightarrow k=\dfrac{1}{\mu}$

We can now rewrite f :

$$f(x)=\begin{cases}0 & x<0\\ \frac{1}{\mu}e^{-x/\mu} & x\geq 0\end{cases},\ \mu>0$$

Ch. 9 Review # 21

The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.

(a) What is the probability that a customer is served in the first 3 minutes?

(b) What is the probability that a customer has to wait more than 10 minutes?

(c) What is the median waiting time?

$$f(x)=\begin{cases}0 & x<0\\ \frac{1}{8}e^{-x/8} & x\geq 0\end{cases},$$

$$(a)\ P(X<3)=\int_{-\infty}^{3} f(x)\,dx=\frac{1}{8}\int_0^3 e^{-x/8}dx=\left[\frac{1}{\cancel{8}}\cdot -\cancel{8}e^{-x/8}\right]_0^3=\left[-e^{-x/8}\right]_0^3=-e^{-3/8}+1$$

$$\approx 0.3127$$

Ch. 9 Review # 21

The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.

(a) What is the probability that a customer is served in the first 3 minutes?

(b) What is the probability that a customer has to wait more than 10 minutes?

(c) What is the median waiting time?

$$(b)\ P(X>10)=\int_{10}^{\infty} f(x)\,dx=\frac{1}{8}\lim_{b\to\infty}\int_{10}^{b} e^{-x/8}dx=\lim_{b\to\infty}\left[-e^{-x/8}\right]_{10}^{b}=\lim_{b\to\infty}\left[\frac{-1}{e^{x/8}}\right]_{10}^{b}$$

$$=\lim_{b\to\infty}\left[\frac{-1}{e^{b/8}}-\frac{-1}{e^{5/4}}\right]=\frac{-1}{\lim\limits_{b\to\infty}e^{b/8}}+\frac{1}{e^{5/4}}=\frac{1}{e^{5/4}}\approx 0.2685$$

$$(c)\ \frac{1}{8}\int_m^{\infty} e^{-x/8}dx=\frac{1}{2}\Rightarrow\frac{1}{8}\lim_{b\to\infty}\int_m^{b} e^{-x/8}dx=\frac{1}{2}\Rightarrow\lim_{b\to\infty}\left[-e^{-x/8}\right]_m^b=\frac{1}{2}\Rightarrow\lim_{b\to\infty}\left[\frac{-1}{e^{x/8}}\right]_m^b=\frac{1}{2}$$

$$\Rightarrow\lim_{b\to\infty}\frac{-1}{e^{b/8}}-\frac{-1}{e^{m/8}}=\frac{1}{2}\Rightarrow\frac{1}{e^{m/8}}=\frac{1}{2}\Rightarrow e^{m/8}=2\Rightarrow\frac{m}{8}=\ln 2 \qquad m=8\ln 2$$

$$\approx 5.55\text{ min.}$$

9.5 # 18

The standard deviation for a random variable with probability density function f and mean μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\right]^{1/2}$$

Find the standard deviation for an exponential density function with mean μ.

$$\int (x-\mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = -(x-\mu)^2 e^{-x/\mu} - 2\mu(x-\mu)e^{-x/\mu} - 2\mu^2 e^{-x/\mu} + C$$

Diff		*Int*
$(x-\mu)^2$	$+$	$\frac{1}{\mu}e^{-x/\mu}$
$2(x-\mu)$	$-$	$-e^{-x/\mu}$
2	$+$	$\mu e^{-x/\mu}$
0		$-\mu^2 e^{-x/\mu}$

$$\int (x-\mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = \frac{-(x-\mu)^2}{e^{x/\mu}} - \frac{2\mu(x-\mu)}{e^{x/\mu}} - \frac{2\mu^2}{e^{x/\mu}} + C$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \lim_{b\to\infty}\int_0^b (x-\mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = \lim_{b\to\infty}\left[\frac{-(x-\mu)^2}{e^{x/\mu}} - \frac{2\mu(x-\mu)}{e^{x/\mu}} - \frac{2\mu^2}{e^{x/\mu}}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{-(b-\mu)^2}{e^{b/\mu}} - \frac{2\mu(b-\mu)}{e^{b/\mu}} - \frac{2\mu^2}{e^{b/\mu}}\right] - \left[-(0-\mu)^2 - 2\mu(0-\mu) - 2\mu^2\right]$$

$$= \lim_{b\to\infty}\frac{-\left[(b-\mu)^2 + 2\mu(b-\mu) + 2\mu^2\right]}{e^{b/\mu}} - \left[-\mu^2 + 2\mu^2 - 2\mu^2\right]$$

$$= \lim_{b\to\infty}\frac{-\left[b^2 - \cancel{2b\mu} + \mu^2 + \cancel{2b\mu} - \cancel{2\mu^2} + \cancel{2\mu^2}\right]}{e^{b/\mu}} + \mu^2$$

$$= \lim_{b\to\infty}\frac{-\left[b^2 + \mu^2\right]}{e^{b/\mu}} + \mu^2 \overset{L'H}{=} \lim_{b\to\infty}\frac{-2b}{\frac{1}{\mu}e^{b/\mu}} + \mu^2 \overset{L'H}{=} \lim_{b\to\infty}\overbrace{\frac{-2}{\frac{1}{\mu^2}e^{b/\mu}}}^{0} + \mu^2 = \mu^2$$

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx}$$

$$\boxed{\sigma = \mu}$$

For the Exponential Density Function