7.8 L'Hopital's Rule

$\lim_{x\to a}\frac{f(x)}{g(x)}$ when $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$

$\lim_{x\to a}\frac{f(x)}{g(x)} = "\frac{0}{0}"$ ← this is called an indeterminate form

$\lim_{x\to a}\frac{f(x)}{g(x)}$ when $f(x)\to \pm\infty$ and $g(x)\to \pm\infty$ as $x\to a$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \pm "\frac{\infty}{\infty}"$ ← this is called an indeterminate form

These two types of indeterminate forms can be simplified using

L'Hopital's Rule

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

$$\lim_{x\to 1}\frac{x^5-1}{x^9-1} = "\frac{0}{0}"$$

$$\overset{L'H}{=} \lim_{x\to 1}\frac{5x^4}{9x^8} = \boxed{\frac{5}{9}}$$

$$\lim_{x\to 0}\frac{\sin(4x)}{\tan(5x)} = "\frac{0}{0}"$$

$$\stackrel{L'H}{=} \lim_{x\to 0}\frac{4\cos(4x)}{5\sec^2(5x)}$$

$$= \boxed{\frac{4}{5}}$$

$$\cos 0 = 1$$

$$\sec 0 = \frac{1}{\cos 0} = 1$$

$$\lim_{x\to 0}\frac{x}{\arctan(8x)} = "\frac{0}{0}"$$

$$\stackrel{L'H}{=} \lim_{x\to 0}\frac{1}{\frac{1}{1+(8x)^2}\cdot 8} = \lim_{x\to 0}\frac{1}{\frac{8}{1+64x^2}} = \lim_{x\to 0}\frac{1+64x^2}{8} = \boxed{\frac{1}{8}}$$

$$\lim_{x\to\infty}\frac{(\ln x)^2}{x} = "\frac{\infty}{\infty}"$$

$$\stackrel{L'H}{=} \lim_{x\to\infty}\frac{2\ln x\cdot\left(\frac{1}{x}\right)}{1}$$

$$= \lim_{x\to\infty}\frac{2\ln x}{x} = "\frac{\infty}{\infty}"$$

$$\stackrel{L'H}{=} \lim_{x\to\infty}\frac{2\left(\frac{1}{x}\right)}{1}$$

$$= \lim_{x\to\infty}\frac{2}{x} = \boxed{0}$$

Other indeterminate forms

Need to be converted into $"\frac{0}{0}"$ or $"\frac{\infty}{\infty}"$

1. **Indeterminate products** $"0\cdot\pm\infty"$

Write the product as a quotient $f\cdot g=\dfrac{f}{\frac{1}{g}}$ or $f\cdot g=\dfrac{g}{\frac{1}{f}}$

$$\lim_{x\to\infty} x\tan\left(\frac{4}{x}\right) = "\infty\cdot 0"$$

$$\lim_{x\to\infty} x\tan\left(\frac{4}{x}\right) = \lim_{x\to\infty}\frac{\tan\left(\frac{4}{x}\right)}{\frac{1}{x}} = "\frac{0}{0}"$$

Chain Rule

$$\overset{L'H}{=}\lim_{x\to\infty}\frac{\sec^2\left(\frac{4}{x}\right)\cdot\left(\frac{-4}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} 4\sec^2\left(\frac{4}{x}\right) = \boxed{4}$$

Other indeterminate forms

2. **Indeterminate differences** $"\infty-\infty"$

Convert the difference into a quotient

How?

$a)$ factor out a common factor

$b)$ find a common denominator

$c)$ rationalize

$$\lim_{x\to\infty} xe^{6/x}-x = "\infty-\infty"$$

$$=\lim_{x\to\infty} x\left(e^{6/x}-1\right) = "\infty\cdot 0"$$

$$=\lim_{x\to\infty}\frac{e^{6/x}-1}{\frac{1}{x}} = "\frac{0}{0}" \overset{L'H}{=} \lim_{x\to\infty}\frac{e^{6/x}\left(\frac{-6}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} 6e^{6/x} = \boxed{6}$$

$$\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)="\infty-\infty"$$

Get a common denominator

$$=\lim_{x\to1^+}\left(\frac{x\cdot\ln x}{(x-1)\ln x}-\frac{1(x-1)}{(x-1)\ln x}\right)=\lim_{x\to1^+}\left(\frac{x\cdot\ln x-x+1}{(x-1)\ln x}\right)="\frac{0}{0}"$$

$$\overset{L'H}{=}\lim_{x\to1^+}\frac{1\cdot\ln x+x\cdot\frac{1}{x}-1}{1\cdot\ln x+(x-1)\cdot\frac{1}{x}}$$

$$=\lim_{x\to1^+}\frac{\ln x+\cancel{1-1}}{\ln x+1-\frac{1}{x}}=\lim_{x\to1^+}\frac{\ln x}{\ln x+1-\frac{1}{x}}="\frac{0}{0}"$$

$$\overset{L'H}{=}\lim_{x\to1^+}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\boxed{\frac{1}{2}}$$

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Other indeterminate forms

3. **Indeterminate powers** $"0^0"$ $"\infty^0"$ $"1^\infty"$

Use ln to convert into a quotient

$$\lim_{x\to a}f(x)^{g(x)}$$

$$y=\lim_{x\to a}f(x)^{g(x)}\quad\Rightarrow\ln y=\ln\left[\lim_{x\to a}f(x)^{g(x)}\right]\quad\Rightarrow\ln y=\lim_{x\to a}\left[\ln f(x)^{g(x)}\right]$$

let y equal the limit — take the ln of both sides — interchange ln and lim

$$\Rightarrow\ln y=\lim_{x\to a}\left[g(x)\cdot\ln f(x)\right]\quad\leftarrow\text{this will be in the form }"0\cdot\pm\infty"$$

use the power rule for ln — follow the directions for $0\cdot\infty$ to find the limit L

$$\Rightarrow\ln y=L\quad\Rightarrow e^{\ln y}=e^L\quad\Rightarrow y=e^L$$

make both sides the exponent on e to get back to y

$$\lim_{x\to 0}(1-6x)^{\frac{1}{x}} = \text{"}1^\infty\text{"}$$

$$y = \lim_{x\to 0}(1-6x)^{\frac{1}{x}}$$

$$\ln y = \ln\left[\lim_{x\to 0}(1-6x)^{\frac{1}{x}}\right]$$

$$\ln y = \lim_{x\to 0}\left[\ln(1-6x)^{\frac{1}{x}}\right]$$

$$\ln y = \lim_{x\to 0}\left[\frac{1}{x}\cdot\ln(1-6x)\right]$$

$$\ln y = \lim_{x\to 0}\left[\frac{\ln(1-6x)}{x}\right] = \text{"}\frac{0}{0}\text{"}$$

$$\ln y \overset{L'H}{=} \lim_{x\to 0}\left[\frac{\frac{1}{1-6x}(-6)}{1}\right]$$

$$\ln y = \lim_{x\to 0}\left[\frac{-6}{1-6x}\right]$$

$$\ln y = -6$$

$$e^{\ln y} = e^{-6}$$

$$y = e^{-6}$$

$$\lim_{x\to 0}(1-6x)^{\frac{1}{x}} = \boxed{e^{-6}}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \text{"}1^\infty\text{"}$$ a, b real numbers

$$y = \lim_{x\to\infty}\left(1+\tfrac{a}{x}\right)^{bx}$$

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\tfrac{a}{x}\right)^{bx}\right]$$

$$\ln y = \lim_{x\to\infty}\left[\ln\left(1+\tfrac{a}{x}\right)^{bx}\right]$$

$$\ln y = \lim_{x\to\infty}\left[bx\cdot\ln\left(1+\tfrac{a}{x}\right)\right] = \text{"}\infty\cdot 0\text{"}$$

$$\ln y = \lim_{x\to\infty}\left[\frac{b\cdot\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}\right] = \text{"}\frac{0}{0}\text{"}$$

$$\ln y \overset{L'H}{=} \lim_{x\to\infty}\left[\frac{\frac{b}{1+\frac{a}{x}}\left(\frac{-a}{x^2}\right)}{\frac{-1}{x^2}}\right]$$

$$\ln y = \lim_{x\to\infty}\left[\frac{ab}{1+\frac{a}{x}}\right]$$

$$\ln y = ab$$

$$e^{\ln y} = e^{ab}$$

$$y = e^{ab}$$

$$y = \lim_{x\to\infty}\left(1+\tfrac{a}{x}\right)^{bx} = \boxed{e^{ab}}$$