

Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ (where $b_n > 0$) satisfies:

i) $\lim_{n \rightarrow \infty} b_n = 0$

ii) $\{b_n\}$ is a decreasing sequence

then $|R_n| = |s - s_n| \leq b_{n+1}$

The size of the error is at most _____.

The actual sum is between _____

The error has _____.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} - \frac{1}{100} + \frac{1}{121} - \frac{1}{144} \dots$$

$\underbrace{\hspace{15em}}_{s_9} \qquad \underbrace{\hspace{10em}}_{R_9}$
 $\underbrace{\hspace{25em}}_s$

The error committed in using the 9th partial sum to approximate the total sum is R_9

The size of this error is at most the size of the first omitted term.

$$|R_9| = |s - s_9| \leq \frac{1}{100} \quad \Rightarrow \quad \frac{-1}{100} \leq s - s_9 \leq \frac{1}{100}$$

$$s_9 - \frac{1}{100} \leq s \leq s_9 + \frac{1}{100}$$

The actual sum is between $s_n - b_{n+1}$ and $s_n + b_{n+1}$.

The sign of the error is the sign of the first omitted term.

$$R_9 = s - s_9 < 0 \quad \Rightarrow \quad s_9 > s \quad s_9 \text{ is an overestimate}$$

since $a_{10} = -\frac{1}{100}$

Fall 2011

9. Which of the following is the best approximation of $\ln(\frac{11}{10})$?

- (A) 0 (B) $\frac{1}{10}$ (C) $\frac{5}{100}$ (D) $\frac{9}{100}$ (E) $\frac{95}{1000}$ (F) $\frac{99}{1000}$ (G) $\frac{109}{1000}$ (H) $\frac{155}{1000}$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \text{ with } R=1$$

Taylor Series Estimation Theorem

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I ,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I , and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The Remainder Estimation Theorem If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a , inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n + 1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f , then the series converges to $f(x)$.

Consider the polynomial $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ as an approximation to e^x on the interval $-2 \leq x \leq 2$. What is the best bound on the error for this estimate that is given by Taylor's inequality?

- (a) $1/24$ (b) $e/12$ (c) $2e^2/3$ (d) $e^3/4$ (e) $3e^4/2$ (f) e^5