10.2 Series

We will now add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

to get $\underbrace{a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots}$ <u>Notation:</u>

this is called an infinite _____

Example:

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

S_n = the sum of the first n terms

it is called the _____

$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$

$S_1 =$

The partial sums form a sequence $\{S_n\}_{n=1}^{\infty}$

$S_2 = 2 + \frac{2}{3} =$

$S_3 = 2 + \frac{2}{3} + \frac{2}{9} =$ $\{S_n\}_{n=1}^{\infty} =$

$S_4 = 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} =$

n	S_n
1	2
2	2.66666
3	2.88888
4	2.96296
5	2.98765
10	2.99995
15	2.99999
20	2.99999
25	2.99999

$\lim_{n \to \infty} S_n = s$ $\Rightarrow$ We call s the _____ of the infinite series

(the limit of the
sequence of partial
sums exists and is finite)

$$\sum_{n=1}^{\infty} a_n = s$$

and the series is called _____

(by adding sufficiently many terms of the series, we can get as close as we like to the number s.)

otherwise the series is called _____

> The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ _____.
>
> We will show this in 12.3

$\{S_n\}_{n=1}^{\infty} = \left\{ 2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots \right\}$ It seems like $\lim_{n \to \infty} S_n = 3$

$$\Rightarrow 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = \sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$$

We can show that the sum is 3 since this series is an example

of a special type of series called a _____ series.

A _____ is one in which each term is obtained from the preceding one by multiplying it by the common ratio r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

this only converges _____ .

$\underline{r = 1}$

$S_n = a + a + a + a + \cdots = na \Rightarrow \lim_{n \to \infty} na =$

$\underline{r = -1}$

$S_n = a - a + a - a + \cdots = (-1)^{n-1} a \Rightarrow \lim_{n \to \infty} (-1)^{n-1} a$

$\left(\begin{array}{l}\text{it could be } a, \text{ it could be } 0 \\ \text{depending on the value of } n\end{array}\right)$

$\underline{r \neq \pm 1}$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\left(1 + r + r^2 + r^3 + \cdots + r^{n-1}\right)$$

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n = a\left(r + r^2 + r^3 + \cdots + r^n\right)$$

$S_n - rS_n = \qquad\qquad a\left(1 - r^n\right)$

$\Rightarrow S_n(1 - r) = a\left(1 - r^n\right)$

$\Rightarrow S_n = \dfrac{a\left(1 - r^n\right)}{1 - r}$

so, $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \dfrac{a\left(1 - r^n\right)}{1 - r} = \dfrac{a\left(1 - \lim_{n \to \infty} r^n\right)}{1 - r} = \dfrac{a\left(1 - \overset{=0}{\lim_{n \to \infty} r^n}\right)}{1 - r} = \dfrac{a}{1 - r}$

We saw in section 12.1 : $\qquad\qquad\qquad$ if $-1 < r < 1$

$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$

$\lim_{n \to \infty} r^n = \begin{cases} \text{undefined} & \text{if } r \in (-\infty, -1] \\ \infty & \text{if } r \in (1, \infty) \end{cases}$

so, $\lim_{n \to \infty} S_n = \dfrac{a}{1 - r}$ provided that $-1 < r < 1$ or $|r| < 1$.

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to the sum of $\dfrac{a}{1 - r}$ if $|r| < 1$
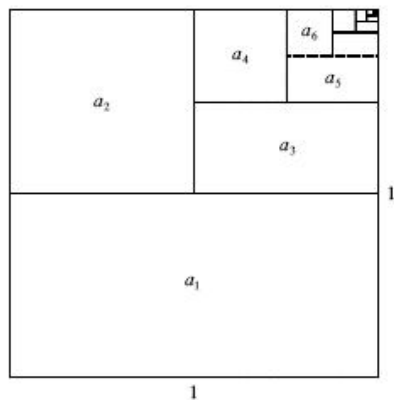
The geometric series diverges for all other values of r

Back to our example:

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = 2\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n-1}} + \cdots\right)$$

$a = $ the first term

$r = $ the ratio b/w the terms

Area of square $= 1$

sum of the series should also be 1

Find $\displaystyle\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

Represent $2.\overline{15}$ as an improper fraction by using a geometric series.

$2.\overline{15} = 2.151515\ldots$

$\qquad = 2 +$

$\qquad$ a geometric series with $a = \underline{\quad}$ and $r = \underline{\quad}$

$$s = \frac{a}{1-r} =$$

A **telescoping series** is one in which the middle terms cancel

and the sum collapses into just a few terms.

Example:

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

$S_n =$

If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then _____.

Converse :

If $\displaystyle\lim_{n\to\infty} a_n = 0,$ then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. This is _____

$\left(\text{Just because } \lim_{n\to\infty} a_n = 0, \text{ you } \rule{1cm}{0.4pt} \text{ conclude that the series } \sum_{n=1}^{\infty} a_n \text{ is convergent.}\right)$

Contrapositive :

Test for Divergence :

$\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)}$

If $\displaystyle\sum a_n$ and $\displaystyle\sum b_n$ are convergent series,

then so are the series $\displaystyle\sum c a_n$ (where c is a constant),

$\displaystyle\sum (a_n + b_n),$ and $\displaystyle\sum (a_n - b_n),$

      i) $\displaystyle\sum c a_n =$

      ii) $\displaystyle\sum (a_n + b_n) =$

      iii) $\displaystyle\sum (a_n - b_n) =$