# 10.3 The Integral Test



If $f(x)$ is: $a)$ continuous, on the interval $[k,\infty)$ constant $k>0$
$b)$ positive,
$c)$ and decreasing

,then the series $\sum_{n=k}^{\infty} a_n$ (with $a_n = f(n)$)

$i)$ is convergent when $\int_k^{\infty} f(x)\,dx$ is convergent.

$ii)$ is divergent when $\int_k^{\infty} f(x)\,dx$ is divergent.

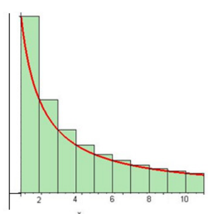
Note:
the function does not necessarily have to be decreasing for all $x \in [k,\infty)$
as long as the function is decreasing "eventually"
(there is some number $N$ so that $f$ is decreasing for all $x > N$)

The next two slides give you a feeling of **how** the integral test works.



$f(x) = \frac{1}{x}$

on $[1,\infty)$




$a)$ continuous,
$b)$ positive,
$c)$ and decreasing

approximate the area $\int_1^{\infty} \frac{1}{x}dx$ with rectangles of width 1 using the left endpoint

$$A \approx 1(1) + 1\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{3}\right) + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

$$A \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ but this is an **overestimate**

$$\Rightarrow \int_1^{\infty} \frac{1}{x}dx < \sum_{n=1}^{\infty} \frac{1}{n}$$

But $\int_1^{\infty} \frac{1}{x}dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}dx = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b = \infty$

The integral $\int_1^{\infty} \frac{1}{x}dx$ diverges and $\int_1^{\infty} \frac{1}{x}dx < \sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{n}$ must also diverge

the **harmonic series**

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10.3 Integral Test

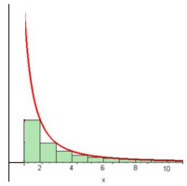
$f(x) = \frac{1}{x^2}$

on $[1, \infty)$

$a)$ continuous,

$b)$ positive,

$c)$ and decreasing



approximate the area $\int_1^{\infty} \frac{1}{x^2} dx$ with rectangles of width 1 using the right endpoint

$$A \approx 1\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{9}\right) + 1\left(\tfrac{1}{16}\right) + \cdots = \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \cdots$$

$$A \approx \sum_{n=2}^{\infty} \frac{1}{n^2}$$ but this is an **underestimate**

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2} dx \quad \Rightarrow 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx$$

But $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \int_1^{b} x^{-2} dx = \lim_{b\to\infty} \frac{-1}{x}\Big|_1^b = \lim_{b\to\infty} \frac{-1}{b} + 1 = 1$

The integral $\int_1^{\infty} \frac{1}{x^2} dx$ converges and $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$

(The sequence of partial sums $S_n$ is a bounded increasing sequence $\Rightarrow$ this sequence converges)

$\Rightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges

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10.3 Integral Test

$f(x) = \frac{1}{x^p}$

on $[1, \infty)$

$a)$ continuous,

$b)$ positive,

$c)$ and decreasing

For what values of $p$ does the integral converge?

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^{b} x^{-p} dx$$

need $-p+1$ to be ________ so that we can get convergence by moving the $x-$term to the ____________

corresponding to this function is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

this is called a __________.

$i)$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges** when ______

$ii)$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **diverges** when _______

Which of these converge?

$a)\ \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ $b)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $c)\ \sum_{n=1}^{\infty} \frac{3}{2n^3}$ $d)\ \sum_{n=1}^{\infty} n^{-e}$

$a)\ \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$

$b)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$c)\ \sum_{n=1}^{\infty} \frac{3}{2n^3}$

$d)\ \sum_{n=1}^{\infty} n^{-e}$



Which of these converge?

$a)\ \sum_{n=1}^{\infty} \frac{1}{n^2+4}$ $b)\ \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $c)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$a)\ \sum_{n=1}^{\infty} \frac{1}{n^2+4}$

so, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ ________ by the ____________.

Which of these converge?



$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $\quad b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $\quad c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$

so, $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ ________ by the ____________.

Which of these converge?



$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $\quad b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $\quad c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

so, $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ ________ by the __________.