10.8 Taylor and Maclaurin Series

Suppose f is a function which has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for $|x-a| < R$

We can find the coefficients c_n in the following manner:

$$f(a) = c_0 + c_1\underbrace{(a-a)}_{0} + c_2\underbrace{(a-a)^2}_{0} + c_3\underbrace{(a-a)^3}_{0} + \cdots \Rightarrow f(a) = c_0$$

Now let's take the derivative:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 + 2c_2\underbrace{(a-a)}_{0} + 3c_3\underbrace{(a-a)^2}_{0} + 4c_4\underbrace{(a-a)^3}_{0} + \cdots \Rightarrow f'(a) = c_1$$

Now let's take the second derivative:

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 + 2\cdot 3c_3\underbrace{(a-a)}_{0} + 3\cdot 4c_4\underbrace{(a-a)^2}_{0} + 4c_5\underbrace{(a-a)^3}_{0} + \cdots$$

$$\Rightarrow f''(a) = 2c_2$$

$$c_2 = \frac{f''(a)}{2}$$

Finally let's take the third derivative:

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots$$

$$f'''(a) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(a-a) + 3\cdot 4\cdot 5c_5(a-a)^2 + \cdots \Rightarrow f'''(a) = 2\cdot 3c_3$$

$$c_3 = \frac{f'''(a)}{2\cdot 3}$$

Continuing in this manner, we'll obtain:

$$c_4 = \frac{f^{(4)}(a)}{2\cdot 3\cdot 4} \qquad c_5 = \frac{f^{(5)}(a)}{2\cdot 3\cdot 4\cdot 5} \qquad \cdots \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

__________ **series** of the function f centered at a.

If $a=0$, then we call the series the ____________ **series** of the function f.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

Find the Maclaurin series for $f(x)=e^x$

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$f(x)=e^x$ $\qquad f(0)=1$

$f'(x)=e^x$ $\qquad f'(0)=1$

$f''(x)=e^x$ $\qquad f''(0)=1$

$f'''(x)=e^x$ $\qquad f'''(0)=1$

$f^{(4)}(x)=e^x$ $\qquad f^{(4)}(0)=1$

$\vdots \qquad \vdots \qquad \vdots$

$$\frac{f^{(n)}(0)}{n!}x^n=$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{(n+1)}\right|=0<1$$

for all x

$\Rightarrow R=\infty$

$e^x=$ $\qquad\qquad$, with $R=\infty$

Find the Maclaurin series for $f(x) = \sin x$

$$f(x) = \sin x \qquad f(0) =$$
$$f'(x) = \cos x \qquad f'(0) =$$
$$f''(x) = -\sin x \qquad f''(0) =$$
$$f'''(x) = -\cos x \qquad f'''(0) =$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) =$$
$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) =$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2(n+1)+1)!}\cdot\frac{x^{\overset{2n+3}{2(n+1)+1}}}{x^{2n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{\cancel{(2n+1)!}}{\underset{2n+3}{(2n+3)(2n+2)}\cdot\cancel{(2n+1)!}}\cdot\frac{\cancel{x^{2n+1}}\cdot x^2}{\cancel{x^{2n+1}}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0 < 1$$
for all x

$$\Rightarrow R = \infty$$

only odd powers so we should use $2n-1$ or $2n+1$

the first term (when $n=0$) is x^1 so we choose $2n+1$

$$\frac{f^{(n)}(0)}{n!}x^n =$$

$\sin x =$, with $R = \infty$

List of important Maclaurin series :

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$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad R = 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{with } R = 1$$

$$\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad \text{with } R = 1$$

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{,with } R = 1$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{,with } R = \infty$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{,with } R = \infty$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{,with } R = \infty$$

Find the third degree Taylor polynomial for $f(x)=\sqrt{x}$ centered at $x=1$.

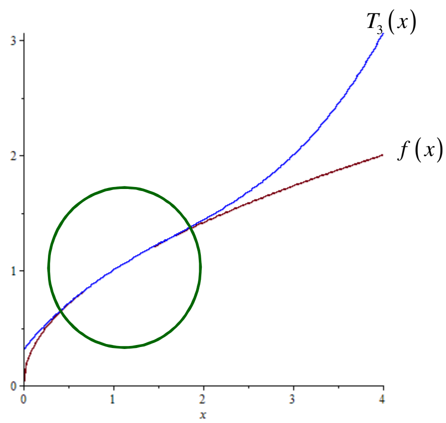
$f(x)=x^{1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}$

$f''(x)=$

$f'''(x)=$

$T_3(x)=$



$T_3(1.5)=1.226562500$

$\sqrt{1.5}\approx 1.224744871$

$T_3(1.5)-\sqrt{1.5}=0.001817629$

10.9/10.10 Applications of Taylor and Maclaurin Series

Use a power series to integrate a function when there is no integration technique you could use.

Integrate and find the coefficient on x^{10}

$\int x^3 e^{-x^3}\,dx$ $\qquad e^x =$

$e^{-x^3} =$

$e^{-x^3} =$

$x^3 e^{-x^3} =$

$\int x^3 e^{-x^3}\,dx =$

Use a power series to evaluate a limit.

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

Use a power series to find the sum of a series.

$$\sum_{n=0}^{\infty}\frac{\pi^n}{n!}=1+\pi+\frac{\pi^2}{2!}+\frac{\pi^3}{3!}+\frac{\pi^4}{4!}+\cdots$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{3!\cdot 8}+\frac{\pi^5}{5!\cdot 32}-\frac{\pi^7}{7!\cdot 128}+\cdots$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots=\frac{\pi}{4}$$

$$4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots\right)=\pi$$

$$\pi=4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\frac{4}{13}-\frac{4}{15}+\cdots$$

This is a series that sums to π but it does it very slowly.

$S_{50}=3.121594653$

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Find the first three non-zero terms of the Maclaurin series for $f(x) = e^x \ln(1-x)$ by __________ two series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \text{ , with } R = \infty \text{ and } I = (-\infty, \infty)$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \text{ , with } R = 1 \text{ and } I = |x| < 1$$

$$e^x \ln(1-x) =$$

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Find the first three non-zero terms of the Maclaurin series for $f(x) = \tan x$ by ________ two series.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \,\Big)\overline{\, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}$$

$$\tan x =$$

$$-\frac{x^3}{6} + \frac{x^3}{2} = \frac{-1+3}{6}x^3 = \frac{x^3}{3}$$

$$\frac{x^5}{120} - \frac{x^5}{24} = \frac{1-5}{120}x^5 = -\frac{x^5}{30}$$

$$-\frac{x^7}{5040} + \frac{x^7}{720} = \frac{-1+7}{5040}x^7 = \frac{x^5}{840}$$

$$-\frac{x^5}{30} + \frac{x^5}{6} = \frac{-1+5}{30}x^5 = \frac{2x^5}{15}$$

$$\frac{x^7}{840} - \frac{x^7}{72} = \frac{3-35}{2520}x^7 = \frac{-4x^7}{315}$$