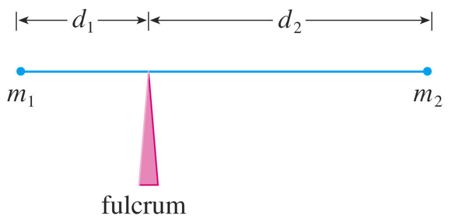
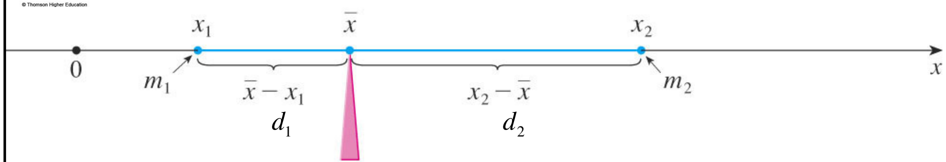
6.6 Center of Mass 1-d



Archimedes' Law of the Lever

the rod will balance if

$$\boxed{m_1 d_1 = m_2 d_2}$$



$$m_1\left(\bar{x} - x_1\right) = m_2\left(x_2 - \bar{x}\right)$$

$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2\bar{x}$$

$$\bar{x}\left(m_1 + m_2\right) = m_1 x_1 + m_2 x_2$$

moment of m_1 (with respect to the origin)

moment of m_2 (with respect to the origin)

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

moment of the system about the origin

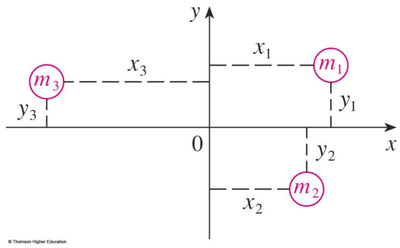
total mass

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\text{total mass}}}$$

$$\underbrace{\bar{x} \cdot \left(\text{total mass}\right)}_{\text{moment for the total mass}} = \underbrace{\sum_{i=1}^{n} m_i x_i}_{\text{moment for the system}}$$

If the total mass was concentrated at $\bar{x}$, then its moment would be the same as the moment for the system.

6.6 Center of Mass 2-d



M_y = moment of the system about the y – axis
measures the tendency of the system
to rotate about the y – axis

$$M_y = m_1x_1 + m_2x_2 + m_3x_3$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$\boxed{\bar{x} = \frac{M_y}{\text{total mass}}}$$

M_x = moment of the system about the x – axis
measures the tendency of the system
to rotate about the x – axis

$$M_x = m_1y_1 + m_2y_2 + m_3y_3$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$\boxed{\bar{y} = \frac{M_x}{\text{total mass}}}$$

$$M_y = \bar{x}(\text{total mass})$$

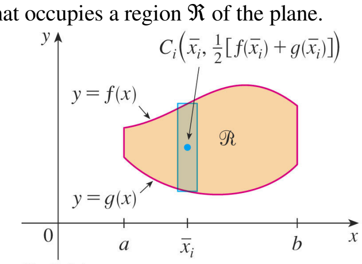
$$M_x = \bar{y}(\text{total mass})$$

The center of mass is the point $(\bar{x}, \bar{y})$ where a single particle with the same mass as the total mass would have the same moments as the system

6.6 Center of Mass 3-d

Consider a flat plate (called a lamina) with uniform density ρ that occupies a region $\mathfrak{R}$ of the plane.

The center of mass of the plate is called the **centroid** of $\mathfrak{R}$.



length $= f(x) - g(x)$
width $= dx$

area $=$ (length)(width)
area $= [f(x) - g(x)]dx$
area $= dA$

$\bar{x}_i$ = average x on the interval

$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}$$

$\bar{y}_i$ = average y on the interval

$$\bar{y}_i = \frac{f(\bar{x}_i) + g(\bar{x}_i)}{2}$$

density $= \rho$ (a constant)

mass $=$ (density)(area)
mass $= \rho[f(x) - g(x)]dx$
mass $= dm$

6.6 Center of Mass 3-d

length $= f(x) - g(x)$ area $= [f(x) - g(x)]dx$ mass $= \rho[f(x) - g(x)]dx$

width $= dx$ area $= dA$ mass $= dm$

<u>Moment about the y-axis</u>

$$M_y = m_1\overline{x_1} + m_2\overline{x_2} + \cdots + m_n\overline{x_n} = \sum_{i=1}^{n} m_i\overline{x_i}$$

let $n \to \infty$ $$M_y = \lim_{n\to\infty}\sum_{i=1}^{n}\overline{x_i}dm = \int x\,dm \Rightarrow \boxed{M_y = \int x\rho[f(x) - g(x)]dx}$$

<u>Moment about the x-axis</u>

$$M_x = m_1\overline{y_1} + m_2\overline{y_2} + \cdots + m_n\overline{y_n} = \sum_{i=1}^{n} m_i\overline{y_i}$$

let $n \to \infty$ $$M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\overline{y_i}dm = \int y\,dm = \int \tfrac{1}{2}[f(x) + g(x)]\rho[f(x) - g(x)]dx$$

$$\Rightarrow \boxed{M_x = \int \tfrac{1}{2}\rho\left[[f(x)]^2 - [g(x)]^2\right]dx}$$

<u>Total mass</u>

$$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{n} m_i = \int dm \Rightarrow \boxed{M = \int \rho[f(x) - g(x)]dx}$$

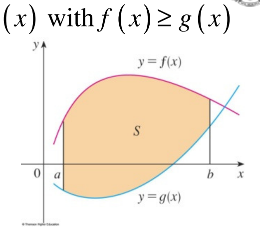
let $n \to \infty$

<u>General Formulas</u>

Math 104 – Rimmer
6.6 Center of Mass

Thin plate : region between $y = f(x)$ and $y = g(x)$ with $f(x) \ge g(x)$

Constant density function $\rho(x) = \rho$



Moment about the y-axis

$$M_y = \rho\int_a^b x\cdot(f(x) - g(x))dx$$

Moment about the x-axis

$$M_x = \frac{\rho}{2}\int_a^b\left([f(x)]^2 - [g(x)]^2\right)dx$$

Mass

$$M = \rho\cdot\int_a^b (f(x) - g(x))dx$$

$$\overline{x} = \frac{M_y}{M} = \frac{\rho\int_a^b x\cdot(f(x) - g(x))dx}{\rho\int_a^b (f(x) - g(x))dx} \Rightarrow \boxed{\overline{x} = \frac{\int_a^b x\cdot(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}}$$

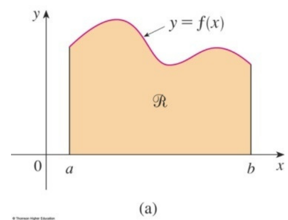
Center of Mass

$(\overline{x}, \overline{y})$

$$\overline{y} = \frac{M_x}{M} = \frac{\frac{\rho}{2}\int_a^b\left([f(x)]^2 - [g(x)]^2\right)dx}{\rho\int_a^b (f(x) - g(x))dx} \Rightarrow \boxed{\overline{y} = \frac{\int_a^b\left([f(x)]^2 - [g(x)]^2\right)dx}{2\cdot\int_a^b (f(x) - g(x))dx}}$$

General Formulas

Thin plate : region under the graph of $y = f(x)$ and above the $x-$axis
Constant density function $\rho(x) = \rho$
Set $g(x) = 0$ in the previous formulas.



(a)

Moment about the $x-axis$

$$M_x = \frac{\rho}{2}\int_a^b \left[f(x)\right]^2 dx$$

Moment about the $y-axis$

$$M_y = \rho\int_a^b xf(x)\,dx$$

Mass

$$M = \rho \cdot \int_a^b f(x)\,dx$$

$$\bar{x} = \frac{M_y}{M} = \frac{\rho\int_a^b x\cdot f(x)\,dx}{\rho\int_a^b f(x)\,dx} \Rightarrow \boxed{\bar{x} = \frac{\int_a^b x\cdot f(x)\,dx}{\int_a^b f(x)\,dx}}$$

Center of Mass

$(\bar{x}, \bar{y})$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{\rho}{2}\int_a^b \left[f(x)\right]^2 dx}{\rho\int_a^b f(x)\,dx} \Rightarrow \boxed{\bar{y} = \frac{\int_a^b \left[f(x)\right]^2 dx}{2\cdot\int_a^b f(x)\,dx}}$$