8.3 Trig. Substitution

1. $\sqrt{a^2 - x^2}$

Let $x = a\sin\theta$

Assume $a > 0$.

$-a \le x \le a$

$x = a\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{a}\right)$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Quadrant 1

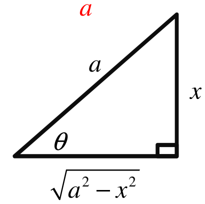
Quadrant 4 but with θ from $\frac{-\pi}{2}$ to 0.

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2}$$
$$= \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2\left(1 - \sin^2\theta\right)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$= \left|a\cos\theta\right|$$

$a > 0$ and $\cos\theta \ge 0$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$ so we can drop the absolute value sign

$$\sqrt{a^2 - x^2} = a\cos\theta$$

$x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta$



Reference Triangle

8.3 Trig. Substitution

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8.3 Trig. Substitution

2. $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 + a^2}$

Let $x = a\tan\theta$

Assume $a > 0$.

$-\infty < x < \infty$

$x = a\tan\theta \Rightarrow \theta = \arctan\left(\frac{x}{a}\right)$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Quadrant 1

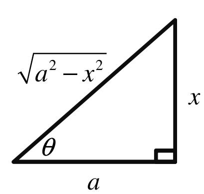
Quadrant 4 but with θ from $\frac{-\pi}{2}$ to 0.

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a\tan\theta)^2}$$
$$= \sqrt{a^2 + a^2\tan^2\theta}$$
$$= \sqrt{a^2\left(1 + \tan^2\theta\right)}$$
$$= \sqrt{a^2\sec^2\theta}$$
$$= \left|a\sec\theta\right|$$

$a > 0$ and $\sec\theta \ge 0$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$ so we can drop the absolute value sign

$$\sqrt{a^2 + x^2} = a\sec\theta$$

$x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta$



Reference Triangle

8.3 Trig. Substitution

3. $\sqrt{x^2-a^2}$

Let $x = a\sec\theta$

Assume $a>0$.

$x<-a$ or $x>a$

$x = a\sec\theta \Rightarrow \theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$

$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

Quadrant 1 Quadrant 2.

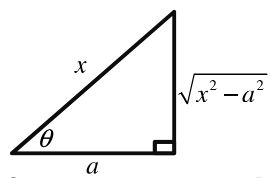
$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2-a^2}$$
$$= \sqrt{a^2\sec^2\theta - a^2}$$
$$= \sqrt{a^2\left(\sec^2\theta-1\right)}$$
$$= \sqrt{a^2\tan^2\theta}$$
$$= |a\tan\theta|$$

$a>0$ but t $\tan\theta \ge 0$ for $0\le\theta<\frac{\pi}{2}$ and $\tan\theta\le 0$ for $\frac{\pi}{2}<\theta\le\pi$
so we can't drop the absolute value sign

$\sqrt{x^2-a^2} = a\tan\theta$ for $x>a$

$\sqrt{x^2-a^2} = -a\tan\theta$ for $x<-a$

$x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta$



Reference Triangle

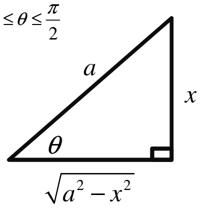
8.3 Trig. Substitution

1. $\sqrt{a^2-x^2}$ Let $x = a\sin\theta$

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2}$$
$$= \sqrt{a^2-a^2\sin^2\theta}$$
$$= \sqrt{a^2\left(1-\sin^2\theta\right)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$= a\cos\theta$$

$x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$



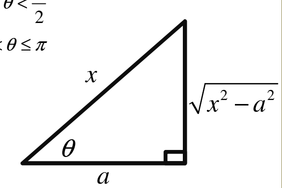
2. $\sqrt{x^2-a^2}$ Let $x = a\sec\theta$

$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2-a^2}$$
$$= \sqrt{a^2\sec^2\theta-a^2}$$
$$= \sqrt{a^2\left(\sec^2\theta-1\right)}$$
$$= \sqrt{a^2\tan^2\theta}$$
$$= a\tan\theta$$

$x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta$

$0 \le \theta < \frac{\pi}{2}$

$\frac{\pi}{2} < \theta \le \pi$



3. $\sqrt{x^2+a^2}$ Let $x = a\tan\theta$

$$\sqrt{x^2+a^2} = \sqrt{(a\tan\theta)^2+a^2}$$
$$= \sqrt{a^2\tan^2\theta-a^2}$$
$$= \sqrt{a^2\left(\tan^2\theta+1\right)}$$
$$= \sqrt{a^2\sec^2\theta}$$
$$= a\sec\theta$$

$x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$