8.4 Partial Fraction Decomposition

Algebra Review:

The **degree** of a polynomial is the highest exponent on x

$$\left.\begin{matrix} x-4 \\ 2x+3 \end{matrix}\right\} \text{linear polynomials} \qquad \left.\begin{matrix} 2x^2-5x-12 \\ x^2-x+3 \end{matrix}\right\} \text{quadratic polynomials}$$

Polynomials that can be factored (over the reals) are called **reducible**.

Polynomials that **can't** be factored (over the reals) are called **irreducible**.

Fundamental Theorem of Algebra

Every polynomial of degree $n > 0$ with real coefficients can be written as a product of linear and/or irreducible quadratic factors.

How can you tell whether ax^2+bx+c is reducible?

$$b^2-4ac \geq 0 \Rightarrow ax^2+bx+c \text{ is reducible}$$

$$b^2-4ac < 0 \Rightarrow ax^2+bx+c \text{ is irreducible}$$

8.4 Partial Fraction Decomposition

Algebra Review:

Reducible

Example 1: $x^2+3x-18$ $\quad b^2-4ac = 9+72 = 81$

When $b^2-4ac > 0$ and is a perfect square, the polynomial should factor nicely because it will have rational roots.

$$x^2+3x-18 = (x+6)(x-3)$$

Example 2: x^2-4x+4 $\quad b^2-4ac = 16-16 = 0$

When $b^2-4ac = 0$, the polynomial will have a double root.

$$x^2-4x+4 = (x-2)(x-2) = (x-2)^2$$

Example 3: $x^2+2x-10$ $\quad b^2-4ac = 4+40 = 44$

When $b^2-4ac > 0$ but not a perfect square, the polynomial doesn't factor nicely because it will have irrational roots. These rarely show up in the context of partial fractions.

Irreducible

Example 4: $x^2-4x+13$ $\quad b^2-4ac = 16-52 = -36$

When $b^2-4ac < 0$, the polynomial does not factor because it will have imaginary roots.

8.4 Partial Fraction Decomposition

Rational Function : $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials

Goal of the Partial Fraction technique: To integrate rational functions.

- ⊙ Write $q(x)$ as a product of linear factors and irreducible quadratic factors.
- ⊙ Use algebra to express the rational function as a sum of simpler fractions.
- ⊙ The simpler fractions should be integrable without too much trouble.

Examples of simpler fractions that can be integrated quickly:

$$\frac{1}{x-4} \qquad \frac{1}{(x-4)^2} \qquad \frac{1}{x^2+4}$$

8.4 Partial Fraction Decomposition

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8.4 Partial Fraction Decomposition

$$\int \frac{1}{x-4}\,dx = \ln|x-4|+C \qquad \begin{aligned} u &= x-4 \\ du &= dx \end{aligned} \qquad \int \frac{1}{u}\,du = \ln|u|+C$$

$$\int \frac{1}{(x-4)^2}\,dx = \frac{-1}{x-4}+C \qquad \begin{aligned} u &= x-4 \\ du &= dx \end{aligned} \qquad \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{-1}{u}+C$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$\frac{1}{x^2+4} = \frac{1}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{4}\cdot\frac{1}{\left(\frac{x}{2}\right)^2+1}$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \qquad \begin{aligned} u &= \tfrac{x}{2} \\ du &= \tfrac{1}{2}\,dx \end{aligned} \Rightarrow 2\,du = dx$$

$$= \frac{1}{4}\int \frac{2}{u^2+1}\,du = \frac{1}{2}\arctan u + C$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C}$$

8.4 Partial Fraction Decomposition

1. The degree of the denominator **must** be greater than the degree of the numerator $\dfrac{p(x)}{q(x)}$
   If it is not, then **long divide** the denominator into the numerator.

2. Decompose the fraction in the following manner: $(A, B, C,$ and D are constants$)$

   i) $q(x)$ can be written as a product of only linear polynomials

$$\frac{5x}{(x-4)(2x+3)} = \frac{A}{x-4} + \frac{B}{2x+3}$$

   ii) $q(x)$ can be written as a product involving powers of linear polynomials

$$\frac{x^2+6x-4}{(x-3)^3(x+5)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{x+5}$$

   iii) $q(x)$ can be written as a product involving irreducible quadratic polynomials

$$\frac{16x-5}{(x^2+2x+10)(x-7)} = \frac{Ax+B}{x^2+2x+10} + \frac{C}{x-7}$$

3. Use algebra to find the constants and then integrate the simpler fractions.