8.7 Improper Integrals

An integral can be called "improper" with one or any combination of the following:

Examples:

- **Infinite upper limit** $\int_1^{\infty} e^{-2x}dx = \lim_{t\to\infty}\int_1^t e^{-2x}dx$
- **Infinite lower limit** $\int_{-\infty}^{1} xe^{x}dx = \lim_{t\to-\infty}\int_t^1 xe^{x}dx$
- **Infinite discontinuity at:**
  - **upper limit** $\int_0^8 \frac{dx}{\sqrt[3]{8-x}} = \lim_{t\to8^-}\int_0^t \frac{dx}{\sqrt[3]{8-x}}$
  - **lower limit** $\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{t\to0^+}\int_t^9 \frac{dx}{\sqrt{x}}$
  - **some value between the upper and lower limit** $\int_{-2}^{3} \frac{dx}{x^4} = \lim_{t\to0^-}\int_{-2}^{t} \frac{dx}{x^4} + \lim_{t\to0^+}\int_t^3 \frac{dx}{x^4}$

If the limit exists, we say the integral converges and if it fails to exist (this includes infinite limits), we say the integral diverges.

8.7 Improper Integrals

The skill for evaluating improper integrals relies on the skills of integration and evaluating limits.

2.6 Limits at Infinity

If $r > 0$ is a rational number, then $\lim_{x\to\infty}\frac{1}{x^r} = 0$

If $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty}\frac{1}{x^r} = 0$

$f(x)$ is a rational function, with **deg. num. = deg. denom.** $\Rightarrow \lim_{x\to\pm\infty} f(x) = \frac{\textbf{coeff. of leading term in num.}}{\textbf{coeff. of leading term in denom.}}$

$f(x)$ is a rational function, with **deg. num. < deg. denom.** $\Rightarrow \lim_{x\to\pm\infty} f(x) = 0$

$f(x)$ is a rational function, with **deg. num. > deg. denom.** $\Rightarrow \lim_{x\to\pm\infty} f(x)$ does not exist (could be ∞ or $-\infty$)

8.7 Improper Integrals

The skill for evaluating improper integrals relies on the skills of integration and evaluating limits.

4.5 L'Hopital's Rule

$\lim_{x\to a}\frac{f(x)}{g(x)}$ when $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$

$\lim_{x\to a}\frac{f(x)}{g(x)}=\text{"}\frac{0}{0}\text{"}$ ← this is called an indeterminate form

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$\lim_{x\to a}\frac{f(x)}{g(x)}$ when $f(x)\to \pm\infty$ and $g(x)\to \pm\infty$ as $x\to a$

$\lim_{x\to a}\frac{f(x)}{g(x)}=\pm\text{"}\frac{\infty}{\infty}\text{"}$ ← this is called an indeterminate form

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These two types of indeterminate forms can be simplified using **L'Hopital's Rule**

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \leftarrow \text{assuming that this limit exists}$$