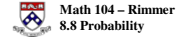


8.8 Probability Density Function



A **random variable**, usually written X , is a variable whose possible values are numerical outcomes of a random phenomenon.

A **continuous random variable** is one which takes an infinite number of possible values (usually measurements)

A **probability density function** is a function f defined for all real x and having the following properties:

1. $f(x) \geq 0$ for all x

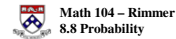
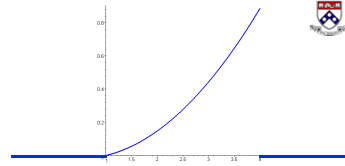
2. $\int_{-\infty}^{\infty} f(x) dx = 1$

» Every continuous random variable, X , has a probability density function.

» Used to determine the probability that a continuous random variable lies between two values

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

$$f(x) = \begin{cases} \frac{2}{27}x(x-1) & 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



a) Verify that f is a probability density function. $f(x) \geq 0$ for all x

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \frac{2}{27} \int_1^4 x(x-1) dx = \frac{2}{27} \int_1^4 (x^2 - x) dx = \frac{2}{27} \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_1^4 \\ &= \frac{2}{27} \left(\left[\frac{64}{3} - 8 \right] - \left[\frac{1}{3} - \frac{1}{2} \right] \right) = \frac{2}{27} \left(21 - 8 + \frac{1}{2} \right) = \frac{2}{27} \left(\frac{27}{2} \right) = 1 \end{aligned}$$

b) Find $P(2 \leq x \leq 3)$.

$$\begin{aligned} P(2 \leq x \leq 3) &= \frac{2}{27} \int_2^3 x(x-1) dx = \frac{2}{27} \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_2^3 \\ &= \frac{2}{27} \left(\left[9 - \frac{9}{2} \right] - \left[\frac{8}{3} - 2 \right] \right) = \frac{2}{27} \left(\frac{66 - 27 - 16}{6} \right) \\ &= \frac{23}{81} \approx 0.284 \end{aligned}$$

Mean (average value) or expected value of a probability density function $f(x)$ is a measure of the center of a pdf.

$$\mu = \int_{-\infty}^{\infty} xf(x) dx$$

Median (m) of a probability density function $f(x)$ is a number such that $\frac{1}{2}$ the area under the graph of f lies to the right of it (and half the area lies to the left of it).

The median m solves the equation

$$\int_m^{\infty} f(x) dx = \frac{1}{2}$$

Variance (σ^2) of a probability density function is a number that is used to measure the spread of a probability density function.

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Standard Deviation (σ) of a probability density function is a better measurement of spread because the units aren't squared.

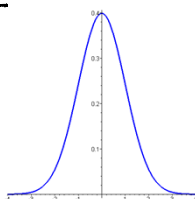
$$\sigma = \sqrt{\sigma^2}$$

Normal Density Function “bell curve”

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ = mean

σ = standard deviation



$\mu = 0$
 $\sigma = 1$ } Standard normal

Z-scores

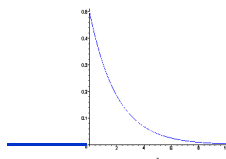
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f(x) = \begin{cases} kx^2 & 2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Find k such that $f(x)$ is a probability density function.

Exponential Density Function

$$f(x) = \begin{cases} 0 & x < 0 \\ ke^{-kx} & x \geq 0 \end{cases}, k > 0$$



Examples of applications:

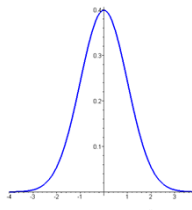
- » life span of electronic components
- » duration of telephone calls
- » waiting time in a doctor's office
- » time b/w successive flight arrivals and departures in an airport

Normal Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$\mu =$$

$$\sigma =$$



Mean (average value) or expected value of a probability density function is a measure of the center of a pdf.

$$\mu =$$

$$f(x) = \begin{cases} \frac{2}{27}x(x-1) & 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean.

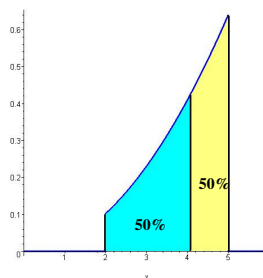
$$f(x) = \begin{cases} 0 & x < 0 \\ ke^{-kx} & x \geq 0, k > 0 \end{cases}$$


Find the mean.

Median (m) of a probability density function is a number such that $\frac{1}{2}$ the area under the graph of f lies to the right of it.

$$f(x) = \begin{cases} \frac{1}{39}x^2 & 2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the median.



Exponential Density Function
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$$f(x) = \begin{cases} 0 & x < 0 \\ ke^{-kx} & x \geq 0, k > 0 \end{cases} \quad \mu = \frac{1}{k} \quad \text{Say } \mu \text{ is known}$$


$$\Rightarrow k = \frac{1}{\mu}$$

We can now rewrite f :

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\mu} e^{-x/\mu} & x \geq 0, \mu > 0 \end{cases}$$

The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.

- What is the probability that a customer is served in the first 3 minutes?
- What is the probability that a customer has to wait more than 10 minutes?
- What is the median waiting time?

 Math 104 – Rimmer
8.8 Probability

The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.

- What is the probability that a customer is served in the first 3 minutes?
- What is the probability that a customer has to wait more than 10 minutes?
- What is the median waiting time?

The standard deviation for a random variable with probability density function f and mean μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \right]^{1/2}$$

Find the standard deviation for an exponential density function with mean μ .

$$\int (x - \mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = -(x - \mu)^2 e^{-x/\mu} - 2\mu(x - \mu) e^{-x/\mu} - 2\mu^2 e^{-x/\mu} + C$$

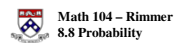
<u>Diff</u>	<u>Int</u>	$\int (x - \mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = \frac{-(x - \mu)^2}{e^{x/\mu}} - \frac{2\mu(x - \mu)}{e^{x/\mu}} - \frac{2\mu^2}{e^{x/\mu}} + C$
$(x - \mu)^2$	$\frac{1}{\mu} e^{-x/\mu}$	
$2(x - \mu)$	$-e^{-x/\mu}$	
2	$\mu e^{-x/\mu}$	
0	$-\mu^2 e^{-x/\mu}$	

$$\begin{aligned} \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx &= \lim_{b \rightarrow \infty} \int_0^b (x - \mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = \lim_{b \rightarrow \infty} \left[\frac{-(x - \mu)^2}{e^{x/\mu}} - \frac{2\mu(x - \mu)}{e^{x/\mu}} - \frac{2\mu^2}{e^{x/\mu}} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{-(b - \mu)^2}{e^{b/\mu}} - \frac{2\mu(b - \mu)}{e^{b/\mu}} - \frac{2\mu^2}{e^{b/\mu}} \right] - \left[\frac{-(0 - \mu)^2}{e^{0/\mu}} - \frac{2\mu(0 - \mu)}{e^{0/\mu}} - \frac{2\mu^2}{e^{0/\mu}} \right] \\ &= \lim_{b \rightarrow \infty} \frac{-(b - \mu)^2 + 2\mu(b - \mu) + 2\mu^2}{e^{b/\mu}} - \left[-\mu^2 + 2\mu^2 - 2\mu^2 \right] \\ &= \lim_{b \rightarrow \infty} \frac{-(b^2 - 2b\mu + \mu^2 + 2b\mu - 2\mu^2 + 2\mu^2)}{e^{b/\mu}} + \mu^2 \\ &= \lim_{b \rightarrow \infty} \frac{-(b^2 + \mu^2)}{e^{b/\mu}} + \mu^2 \stackrel{L'H}{=} \lim_{b \rightarrow \infty} \frac{-2b}{\frac{1}{\mu} e^{b/\mu}} + \mu^2 \stackrel{L'H}{=} \lim_{b \rightarrow \infty} \frac{0}{\frac{1}{\mu^2} e^{b/\mu}} + \mu^2 = \mu^2 \end{aligned}$$

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx}$$

$\sigma = \mu$ For the Exponential Density Function

$$f(x) = \begin{cases} \frac{3}{32}(4 - x^2) & -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



Find the standard deviation σ .