Oral Exam Problems

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May 6, 2010

Algebraic number theory:

Florian: For which prime p is 10 a square mod p? (I gave the congruence conditions mod 40). What's the connection with the prime splitting behavior within the extension $\mathbb{Q}(\sqrt{10})$? How about a composite number n, when is 10 a square mod n? (given the prime factorization of n. I don't know if one can find an efficient way to determine that completely if not given the factorization though, but that part was not required on the test.)

*Can you say something about the density of such prime, and such composite n?

Algebraic topology:

Jonathan: What can you say about the homology and cohomology of a simply connected (which imply orientable) 4-manifold?

NT:

Ted: Back to the splitting problem, when is 10 a fourth power mod a prime p? (This is just asking about the fourth power, not about the higher reciprocity law which is harder.) What is the Galois group of the Galois closure of $\mathbb{Q}(\sqrt[4]{10})$? (A follow up question could be about the density, which is Chebotarev+which element intersect with the Galois subgroup.)

Can you construct some Dihedral extension of order 8 above a local field $\mathbb{Q}_p$? Assume p is not 2. (First identify the possible normal subgroups and their quotients of D_8, then consider the higher ramification groups and use CFT.)

AT:

Jonathon: Can $\mathbb{CP}^2$ be an oriented boundary? What is the homotopy type of $\Omega SO(3)$? How about ΩM_g for the Riemann surfaces?

NT:

Florian: For the dihedral extension problem, can you say something about p=2? Can you explain some of your claims?

Supplement: practise problems: (other than the ones that are similar above)

Ted:

1. Count the number of S^3 extensions of Q_p using CFT and Kummer theory and compare them.

2. Given a biquadratic extension, say something about the unit group. Especially, how about the index inside the unit group of the biquadratic field of the unit group coming from the quadratic sub extensions? After that, how about in an S^3 extension? (use the functional equation to simply the residue-class number-regulator formula, and find the Brauer relations.)

3. Can you say something about two number fields having the same zeta function? Give an example about a number field, whose L functions associated to different irreducible characters are not multiplicatively independent? (Hint: find two subgroups which are potentially Gassman conjugate but not Gassman conjugate.)

4. Higher ramification groups of $\mathbb{Q}_p(\zeta_{p^n})$.

Florian:

1. What are the representatives in Q_p^*/Q_p^{*2}? Compute the prime splitting behavior in some number fields, discriminats, differents, Minkowski bound etc.

2. Show the local-global principle failed for the generalized Fermat curve $3x^3 + 4y^3 + 5z^3 = 0$. (For local case, one has the Hasse Weil bound. For global case, it turns out hard to consider the same process as showing the first/second case of cubic Fermat. One needs some knowledge about elliptic curves. For this example, see Cassel's notes.)

Jonathon:

1. Understand Gysin sequence, it is going to allow you to compute (essentially all of my examples) of bundles. (Later I realized e.g. I can distinguish the total space of circle bundles on all the Riemann surface by Gysin sequence. And it can be used to show the only bundles between three spheres are the Hopf bundles and quaternion/octernion analogue, assuming some famous result.)

2. Lefshetz fixed theorem, can $\mathbb{CP}^2$ cover something? What is the Lefshetz number of the conjugation map on $\mathbb{CP}^2$? Know Alexandra duality.

3. Find a covering space without Deck transfermation. When can a Rieman surface cover another? (my question: what is the universal cover of non orientable closed surfaces?)

4. Prove the Euler characteriztic is even for boundaries.(can use both geometric/SW approach.)

5. Show an example why the homotopy excision failed for high dimension.

6. What is the fundamental group of an infinite genus Riemann surface?

7. A space having the homology as the wedge of two spheres, homopoty type?

8. Prove every codimension one homology class could be represented by a sub manifold. (After getting the sub manifold, to show you really get the homology class you want, identify submanifold with Thom class of normal bundle and then Thom class pulls back.) And codimension two.(use Euler class and transversal intersection of a section of the bundle/another way: functoraility of H^2 and consider inverse image of some subcomplex of maps to $\mathbb{CP}^\infty$.)

9. Tangent bundle on even spheres having no sub bundles.