MATH 210, PROBLEM SET 3

DUE IN LECTURE ON TUESDAY, MARCH 3, AT THE TIME OF THE FIRST MID-TERM.

1. **A multi-option two-person game.**

The object of this set of exercises is to find the optimal strategy in a two-person zero sum game in which the first player has two options and the second has three options. We will find the best strategy for player I in two ways. The first method maximizes the minimum of three linear forms. The second method uses the linear programming method described in class.

The two players in this game are political opponents. An advisor of Player II has just come out with a book claiming that during a phone call, Player II strong armed a foreign country to help him in an election. The two players have to decide how to treat this in the hundreds of ads they will run before an election.

Player I can run two kinds of ads:

1. An ad consisting of a clip of the advisor describing how Player II abused their powers, or
2. An ad which talks about the health care.

Player II can run three kinds of ads:

1. An ad which ignores the incident entirely.
2. An ad which says the phone call was "perfect".
3. An ad which counterattacks by running a doctored Facebook video of the advisor advocating a coup by the "deep state".

Player I wants to choose the relative proportions of the two kinds of ads they run so as to maximize the expected benefit of the ads against the best counterstrategy of Player II.

The payoff matrix for Player I is as follows:

	Player II option 1 "ignore"	Player II option 2 "perfect call"	Player II option 3 "deep state"
Player I option 1 run advisor clip	8	6	-6
Player I option 2 run health care ad	0	1	2

The payoffs to player II are the negatives of the payoffs to player I.

Problems:

1. Based on this payoff matrix, does the "deep state" counterattack work against a pure strategy by player I of always running the clip of the advisor? How would you explain the payoff when player I does not run the clip of the advisor but player II brings up the idea that the advisor advocates a coup by the "deep state"?

2. Does one strategy for either player dominate another strategy by the same player? (Note that if this happens, one can reduce the complexity of the game by eliminating dominated strategies.)

3. Suppose that player I chooses their option #1 with frequency p and their option #2 with frequency $1 - p$. We discussed in class why the minimal payoff to player I of this strategy against any strategy by player # 2 occurs against some pure strategy by player II. Explain why this means that player I should choose p to be a number in the interval $0 \le p \le 1$ where the function

$$h(p) = \min\{8p, 6p + 1 - p, -6p + 2(1 - p)\} \tag{1.1}$$

attains its maximum. Draw the graphs of the three linear functions appearing on the right side of (1.1). Find what value of p player I should pick and explain your reasoning.

4. We are now going to find the solution a second way using linear programming. The payoff matrix is:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} = \begin{pmatrix} 8 & 6 & -6 \\ 0 & 1 & 2 \end{pmatrix}.$$

Recall that the first step in this method is to add a positive constant δ to every entry of A in order to obtain a matrix with all positive entries. Choosing the constant to be $\delta = 7$, one gets the matrix

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix} = \begin{pmatrix} 15 & 13 & 1 \\ 7 & 8 & 9 \end{pmatrix}.$$

The associated linear programming problem is to find a vector $s = (s_1, s_2) \ge (0, 0)$ such that $sB \ge (1, 1, 1)$ and $f(s) = s_1 + s_2$ is as small as possible. Write down

$$s \ge (0, 0) \quad \text{and} \quad sB = (s_1, s_2) \cdot \begin{pmatrix} 15 & 13 & 1 \\ 7 & 8 & 9 \end{pmatrix} \ge (1, 1, 1)$$

as a system of five linear inequalities in the unknowns s_1 and s_2.

5. Continuing with the notation in problem #4, there are now five inequalities to be satisfied by s_1 and s_2. Draw the lines in the s_1-s_2 plane which form the borders of the allowed region defined by these inequalities. Explain why there must be a solution to the linear programming problem in #4 which is an intersection point of two of the 5 lines above. How many of these intersections describe vertices which are in the allowed region? (Hint: On your graph of the lines associated to the inequalities, write down the coordinates of the intersection points of the lines. Then test these points to see if they satisfy all the inequalities in the set of 5 constaints. This will check whether your picture of the lines is correct - an alternative is to check it with Maple!)

Comment: The general fact behind this problem is that every linear programming problem must have a solution which is a **vertex**, in the following sense. A vertex is a point s which satisfies all of the inequalities of a linear programming problem, and for which s is the **unique** vector for which some subset of the inequalities are in fact equalities. When $s = (s_1, s_2)$ is a vector in the plane, the set of inequalities which must hold as equalities if s is to be a vertex must include the equations of two intersecting lines.

6. For each of the vertices you find in problem #5 which are in the allowed region, calculate the objective function

$$f(s) = s_1 + s_2.$$

Determine which of these $s = (s_1, s_2)$ minimizes $f(s)$; these form the set $\mathcal{O}$ of optimal vertices. The set of all solutions s is the span $\mathrm{Span}(\mathcal{O})$ of $\mathcal{O}$. Describe this span. For each s in $\mathrm{Span}(\mathcal{O})$, check that the vector

$$p^* = (p, 1-p) = \frac{1}{f(s)} s = \left(\frac{s_1}{f(s)}, \frac{s_2}{f(s)} \right)$$

gives the same optimal strategy for player I found in problem #3.

Extra Credit

This problem can be turned in at any time during the semester

EC1. Show that in any two-person zero sum game in which Player I has two options and Player II has three options, the optimal strategy $q^* = (q_1, q_2, q_3)$ for player II has the property that one of q_1, q_2 or q_3 is 0. (Hint: Consider the linear programming method for finding $q^* = t/g(t)$, where $t = (t_1, t_2, t_3)$ and $g(t) = t_1 + t_2 + t_3$. The vertices of the allowed region must be the **unique** solution to some system of equations defined by the planes which bound the region. How many of these planes are there, and how many are coordinate planes defined by setting some t_j equal to 0?)