MATH 240: HOMEWORK #3

DUE IN FLORA'S MAILBOX BY NOON ON NOV. 8.

1. The impact of an event on three candidates

Suppose we label three presidential candidates T, B and W. The level of enthusiasm by any one person for these candidates at a given time is a column vector

$$x = \begin{pmatrix} x_T \\ x_B \\ x_W \end{pmatrix}$$

of real numbers.

1. An event in the news has the effect of changing each x to some new vector $T(x)$. This defines a function $T : \mathbb{R}^3 \to \mathbb{R}^3$. Describe in words the significance of saying that $T(\underline{0}) = \underline{0}$ when $\underline{0}$ is the zero vector. Describe in words the significance of saying that $T(rx) = rT(x)$ for all $r \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

2. Suppose x and x' represent the approval vectors associated to two people before the event occurs. Explain why we can view $(x + x')/2$ as the average approval levels of these two people before the event. How would you interpret in words the condition that

$$T((x + x')/2) = (T(x) + T(x'))/2 \tag{1.1}$$

   given that $T(x)$ and $T(x')$ describe the approval levels of the two people after the event?

3. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is continuous, $T(\underline{0}) = \underline{0}$ and that (1.1) holds for all $x, x' \in \mathbb{R}^3$. Show that T must be linear. How would you explain in words the significance of this fact, given your answer to problems # 1 and #2?

   Hints: Use these steps to show T is linear.
   a. Show

$$T(nx) = nT(x) \tag{1.2}$$

   for all x and all integers $n \geq 0$ using induction on n.
   Details: Recall that to prove something by induction, you first check the statement for all n in the range $0 \leq n \leq N$ for some integer N. You then complete the induction by show for $n \geq N$ that if the statement is true for n, it is true for $n + 1$. In this case first check (1.2) for $n = 0$ and $n = 1$. Then use

$$(n + 1)x = nx + x = (n(2x) + (2x))/2. \tag{1.3}$$

   with $n = 0$ to show $T(2x) = 2T(x)$, i.e. that (1.2) holds when $n = 2$. Then use this and (1.3) to show that if (1.3) is true for some $n \geq 2$ it is also true for $n + 1$.
   b. Show $T(-x) = -T(x)$ using $(x + (-x))/2 = \underline{0}$, and conclude that $T(nx) = nT(x)$ for all x and all integers n.
   c. Show $T(rx) = rT(x)$ for all rationals $r = n/m$ and all x, using that $rx = n(x/m)$ and $x = m \cdot (x/m)$.
   d. Use the continuity of T to show that $T(rx) = rT(x)$ for all real numbers r. Show finally that this and (1.1) are enough to prove that T is linear.

4. From now on we suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear. Let the matrix of T be

$$M = \begin{pmatrix} a_{T,T} & a_{T,B} & a_{T,W} \\ a_{B,T} & a_{B,B} & a_{B,W} \\ a_{W,T} & a_{W,B} & a_{W,W} \end{pmatrix}$$

where the entries are real numbers. Explain why $a_{T,B}$ is the partial derivative with respect to enthusiam for candidate B of the change the event brings about in enthusiasm for candidate T. (The other entries have similar interpretations, but you do not have to write these down.)

5. Let $p_M(t) = \det(tI_3 - M)$ be the characteristic polynomial of M, where I_3 is the three by three identity matrix and t is a variable. Using that M has real entries, show that $p_M(t)$ must either have all real roots or one real root r_1 and a pair $\{\sigma, \overline{\sigma}\}$ of complex conjugate non-real roots.

6. Suppose that

$$x = \begin{pmatrix} x_T \\ x_B \\ x_W \end{pmatrix}$$

is the enthusiasm vector of a particular person. Describe in words the significance of x being eigenvector for T with real eigenvalue λ. What does this say about what happens to the views of x after the event that is associated to T? What is the significance of the sign of λ? Must x have real entries?

7. Suppose M has three distinct eigenvalues r_1, r_2 and r_3. Some of these eigenvalues could be non-real complex numbers. We've shown in class that then there is a basis $\{v_1, v_2, v_3\}$ of $\mathbb{C}^3$ considering of of eigenvectors for M. Using your answers to problems #5 and #6, describe conditions on r_1, r_2 and r_3 which are equivalent to there being a basis of eigenvectors of M which is contained in $\mathbb{R}^3$.

8. Suppose now that $p_M(t)$ has a real eigenvector r_1 and two non-real complex conjugate eigenvalues $\{\sigma, \overline{\sigma}\}$. Write $\sigma = re^{i\theta} = r(\cos(\theta) + i\,\sin(\theta))$ for some real number $r > 0$ and some real number θ. Show that there is a basis $B = \{b_1, b_2, b_3\}$ for $\mathbb{R}^3$ such that the matrix of T relative to B has the form

$$[T]_B^B = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r\cos(\theta) & r\sin(\theta) \\ 0 & -r\sin(\theta) & r\cos(\theta) \end{pmatrix} \tag{1.4}$$

Describe in words what the event T does to voters with preferences b_1, b_2 and b_3 when $r_1 = r = 1$ and $\theta = -\pi/2$.

Details: Suppose v_2 is a vector in $\mathbb{C}^3$ which is an eigenvector for T with eigenvalue σ. Write

$$v_2 = x_2 + ix_3$$

for some real vectors $x_2, x_3 \in \mathbb{R}^3$. Use that

$$T(v_2) = T(x_2) + iT(x_3) = \sigma \cdot v_2$$

to calculate $T(x_2)$ and $T(x_3)$. Then show that if x_1 is an eigenvector with eigenvalue r_1, we can take $B = \{x_1, x_2, x_3\}$. To check these vectors are independent, you might first show that $x_2 - ix_3$ is an eigenvector with eigenvalue $\overline{\sigma}$ and that $x_1, x_2 + ix_3, x_2 - ix_3$ are independent over $\mathbb{C}$.

2. Extra Credit

Suppose that $n \geq 1$ and that M is an arbitrary $n \times n$ matrix with real entries. Suppose that the characteristic polynomial $p_M(t) = \det(tI_n - M)$ has m distinct real roots $r_1, \ldots, r_m$ and $n - m = 2\ell$ distinct non-real roots $\sigma_1, \overline{\sigma}_1, \ldots, \sigma_\ell, \overline{\sigma}_\ell$. Here the non-real roots must occur in complex conjugate pairs because $p_M(t)$ has real coefficients. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation represented by M in the standard basis, so that $T(x) = Mx$ for all column vectors x of size n. Show that there is a B basis for $\mathbb{R}^n$ such that the matrix $M' = [T]_B^B$ of T relative to B has the following form. Write

$$\sigma_j = z_j \cdot e^{i\theta_j} = z_j(\cos(\theta_j) + i\,\sin(\theta_j))$$

for some real constants $z_j > 0$ and θ_j. Then M' is a block matrix which has m one-by-one blocks going down the diagonal, with the numbers $r_1, \ldots, r_m$ in these blocks, followed by ℓ two-by-two blocks going down the diagonal which have the form

$$\begin{pmatrix} z_j \cos(\theta_j) & z_j \sin(\theta_j) \\ -z_j \sin(\theta_j) & z_j \cos(\theta_j) \end{pmatrix} \tag{2.5}$$

as j ranges from 1 to ℓ. This generalizes the case $n = 3$ dealt with in the previous problems.