MATH 240: HOMEWORK #5

DUE AT THE FINAL EXAM

1. Linear systems of differential equations and economics

In section 9.7 of the course text, some physical examples of systems of linear differential equations are described which involve systems of springs and chemical solutions flowing between tanks. This homework is about some examples of systems of linear differential equations having to do with the flow of wealth between rich and poor in society. This involves a toy model which is far from definitive. But even toy models lead to asking some relevant questions, e.g. how much does helping the rich also help the poor?

2. The model

Let t stand for time. Suppose that $P(t)$ represents the average net wealth per person of the bottom 10% of the U.S. population, and that $R(t)$ represents the average net wealth per person of the top 10% of the population. Each of these groups interacts with the other group as well as with the rest of the financial system. The rest of the system includes everyone not in one of the groups, the government, and so on. We will assume that the effect of all such interactions is that the rate of change of each of $P(t)$ and of $R(t)$ is the sum of constant functions with functions that are linear in $P(t)$ and $R(t)$.

1. Explain why the above assumptions lead to a differential equation for $y(t) = \begin{pmatrix} R(t) \\ P(t) \end{pmatrix}$ of the form

$$y'(t) = A\ y(t) + V \tag{2.1}$$

in which

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.2}$$

for some real constants a, b, c, d, v_1, v_2.

2. View $R = R(t)$ and $P = P(t)$ as independent variables. Explain why the matrix entry c in the bottom left corner of the matrix A in problem #1 represents the rate of change of $P'(t)$ with respect to $R(t)$. Why can one view this as the partial derivative of the rate at which the wealth of the poor is increasing with respect to the wealth of the rich? The value of c, and even its sign, is a matter of intense debate. Given that people cannot even agree on its sign, we will suppose in what follows that $c = 0$.

3. Explain why the constant a in the upper left corner of A represents the partial derivative of the rate at which the wealth of the rich is changing with respect to the wealth of the rich. It is not very controversial to suppose a is positive. Show that by rescaling the time variable t to nt for some positive constant n, we can assume that $a = 1$. Thus in what follows, we will assume

$$A = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.3}$$

4. How would you describe in words the significance of assuming that $c = b = 0$? What is the general solution of the system (2.1) when $c = b = 0$? What is the significance of the size of d in comparison to 1 in this case, e.g. the meaning of $d < 1$, $d = 1$ and of $d > 1$?

5. Suppose now that $b \neq 0 = c$ and that $0 < d < 1 = a$. Describe in words the significance of assuming $b > 0$ and of assuming $b < 0$. What is a pair $Y_1(t) = \begin{pmatrix} y_{1,1}(t) \\ y_{2,1}(t) \end{pmatrix}$ and $Y_2(t) = \begin{pmatrix} y_{1,2}(t) \\ y_{2,2}(t) \end{pmatrix}$ of independent solutions of the homogenous linear system

$$y'(t) = A\, y(t) \tag{2.4}$$

when we assume $0 < d < 1 = a$ and $b \neq 0 = c$?

6. If we drop all assumptions about the real number entries of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is it possible that there is a periodic solution $y(t)$ to the homogeneous equation (2.4)? Why or why not?