MATH 702: HOMEWORK #5

DUE WEDNESDAY, DEC. 19, 2012.

Please send to ted@math.upenn.edu either a .pdf file of your solutions, either by writing them up in tex or by using the department's scanner to scan handwritten homeworks. Janet Burns in the math department office can help with scanning homeworks.

1. Dedekind subrings of global function fields.

Suppose F is a global function field, i.e. a finite separable extension of $k(t)$ for some finite field k. The abstract curve C_F associated to F is the set of discrete valuations $v : F^* \to \mathbb{Z}$ which are trivial on k, the latter condition holding automatically in this case because k^* is a finite group. Suppose S is a non-empty subset of C_F. In class we defined the S-integers of F to be the ring

$$R(F,S) = \{0\} \cup \{\alpha \in F^* : v(\alpha) \geq 0 \text{ for } v \notin S\}.$$

If $f \in F^*$, $v \in C_F$ and $v(f) < 0$ we will say v is a pole of f; if $v(f) > 0$ then v is a zero of f. In class we used the Riemann Roch theorem to show that $R(F,S)$ has fraction field F. (That involved showing that for every function $f \in F$ there is a function $0 \neq g \in F$ which has its only poles in S and a high order zero at every pole of f; then $g, fg \in R(F,S)$.) Finish the proof that $R(F,S)$ is a Dedekind ring by the following steps.

1. Check that $R(F,S)$ is integrally closed in F.
2. The remaining issue is to show that every non-zero prime ideal P of $R(F,S)$ is maximal. Supppose $0 \neq f \in P$. Show that it will suffice to show that if J is the principal $R(F,S)$-ideal $R(F,S) \cdot f$, then $R(F,S)/J$ is a finite dimensional vector space over the finite field k.
3. Let f be as in problem # 2. In class we checked that the principal divisor $\text{div}(f) = \sum_{v \in C_F} v(f)v$ is well defined, i.e. that $v(f) = 0$ for almost all v. Define $T(f) = \sum_{v \in C_F, v \notin S} v(f)v$, where $v(f) \geq 0$ for $v \notin S$ by the definition of $f \in R(F,S)$. Let $\underline{0}$ be the trivial divisor, and suppose T is a divisor such that $\underline{0} \leq T \leq T(f)$. Define
$$R(F,S,T) = \{0\} \cup \{\alpha \in R(F,S) : \sum_{v \in C, v \notin S} v(\alpha)v \geq T\}.$$
Show that $R(F,S,\underline{0}) = R(F,S)$ and $R(F,S,T(f)) \subset R(F,S)f$. Conclude using problem #2 that it will suffice to show $R(F,S)/R(F,S,T(f))$ is finite dimensional over k.
4. Show that if T is a divisor as in problem # 3 and $v \in C_F - S$, then $R(F,S,T+v) \subset R(F,S,T)$. Then use a uniformizing parameter at v to show that there is a k-linear injection from $R(F,S,T)/R(F,S,T+v)$ into the residue field $k(v)$ of v. Use this and problems #2 and #3 to finish the proof that $R(F,S)$ is Dedekind.

2. The Minkowski bound and computations of units

5. In class we showed that the field $K = \mathbb{Q}(\theta)$ generated by a root θ of $x^3 - x - 1 = 0$ has ring of integers $\mathbb{Z}[\theta]$. This was done by computing the discriminant of $\mathbb{Z}[\theta]$ over $\mathbb{Z}$ and by noting that the this must be the square of an integer times the disciminant of O_K over $\mathbb{Z}$. Show that K has class number 1.
6. The Dirichlet unit theorem says that if L is a number field with ring of integers O_L, then the rank of the unit group O_L^* is $r_1(L) + r_2(L) - 1$. For K as in problem # 5, show that the unit group of O_K is $\{\pm\theta^n\}_{n=1}^{\infty}$. (Hint: If $\theta = u^m$ for some $u \in O_K^*$ and $m > 1$, consider the norm to $\mathbb{Q}$ of $u - 1$.)

7. Show that $K = \mathbb{Q}(\sqrt{30})$ has class number two and unit group $O_K^* = \{\pm(11 + 2\sqrt{30})^n\}_{n\in\mathbb{Z}}$. (Hint: Compute some norms to produce relations between ideal classes of small prime ideals.)

3. The Minkowski bound over function fields.

Suppose q is a prime power, and let $\mathbb{F}_q$ be a finite field of order q. Let $L = \mathbb{F}_q(t)$ be the rational function field in one variable t over $\mathbb{F}_q$. Define v_∞ to be the discrete valuation on L such that $v_\infty(g(t)) = -\deg(g(t))$ for $0 \neq g(t) \in \mathbb{F}_q[t]$. Suppose F is a finite separable extension of L. Define $n = [F : L]$. For simplicity, we will assume that v_∞ splits completely in F, in the sense that there are n distinct discrete valuations $w_1, \ldots, w_n$ on F which extend v_∞. Such F are analogs of finite totally real extensions of $\mathbb{Q}$. This problem is about a variation on the Minkowski method for bounding the class number of the integral closure A of $\mathbb{F}_q[t]$ in F.

8. Let $O_{v_\infty} \subset L$ be the valuation ring of v_∞. The valuations $w_1, \ldots, w_n$ correspond to distinct prime ideals of the integral closure of O_{v_∞} in F. Show that the inclusion $L \subset F$ gives rise to an isomorphism of completions $L_{v_\infty} \to F_{w_i}$ for all $i = 1, \ldots, n$. This identifies F_{w_i} with the formal power series field $L_{v_\infty} = \mathbb{F}_q((t^{-1}))$. The valuation ring O_{w_i} of F_{w_i} is thus identified with $\mathbb{F}_q[[t^{-1}]]$.
9. Define a Haar measure μ_∞ on $L_{v_\infty} = \mathbb{F}_q((t^{-1}))$ by the requirement that $\mu_\infty(\mathbb{F}_q[[t^{-1}]]) = 1$. Thus $\mu(a + t^{-b}\mathbb{F}_q[[t^{-1}]]) = q^{-b}$ for $a \in L_{v_\infty}$ and $b \in \mathbb{Z}$, since μ is invariant under translation and additive over unions of disjoint open subsets. Define μ to be the Haar measure on $\prod_{i=1}^n F_{w_i}$ which is the product of μ_∞ on each factor $L_{w_i} \equiv \mathbb{F}_q((t^{-1}))$. Suppose $C = (c_{i,j})$ is an invertible $n \times n$ matrix whose entries $c_{i,j}$ lie in $\mathbb{F}_q((t^{-1}))$. Show that if $U = \prod_{i=1}^n O_{w_i}$ then $\mu(O_{w_i}) = 1$. Using the identifications in problem 8, show that the image $C \cdot U$ of U under left multiplication by C is a compact open subset of $\prod_{i=1}^n F_{w_i}$ with
$$\mu(C \cdot U) = q^{-v_\infty(\det(C))}$$
Hints: U and $C \cdot U$ are finitely generated free modules of rank n for the discrete valuation ring $\mathbb{F}_q[[t^{-1}]] = B$. By multiplying C by a non-zero scalar which is close to 0 in B show that it is enough to consider the case in which C has entries in B and $C \cdot U \subset U$. Show that the map $U \to U$ given by $u \to Cu$ induces multiplication by $\det(C)$ on the top exterior power $\Lambda^n U$ of U over B. Then compute $\det(C)B$ a different way by applying the fundamental theorem about finitely generated modules over a P.I.D. to the inclusion of free B-modules of rank n given by $C \cdot U \subset U$.
10. Identify each F_{w_i} with $\mathbb{F}_q((t^{-1}))$ as in problem 8. Show that with this identification, $X = \oplus_{i=1}^n \mathbb{F}_q[t]$ is a discrete subgroup of $Y = \oplus_{i=1}^n F_{w_i}$, in the sense that there is an open neighborhood of each element of X which contains no other element of X. Show that if U is as in problem 9, then $t^{-1}U = \oplus_{i=1}^n t^{-1}O_{w_i}$ is a fundamental domain for X, in the sense that the inclusion $t^{-1}U \to Y$ gives a topological isomorphism $t^{-1}U \to Y/X$. Conclude that $\mu(Y/X) = \mu(t^{-1}U) = q^{-n}$, i.e. X has covolume q^{-n} in Y.
11. Explain why the integral closure A of $\mathbb{F}_q[t]$ in F is a finitely generated free $\mathbb{F}_q[t]$ module of rank n. Let $a_1, \ldots, a_n$ be generators for this module. Let $\psi : F \to \oplus_{i=1}^n F_{w_i}$ be the natural homomorphism. Define C to be the $n \times n$ matrix $(c_{i,j})$ such that the j^{th} column is the vector $\psi(a_j)$ considered as a column vector when we identify each F_{w_i} with $\mathbb{F}_q((t^{-1}))$. Explain why the elements of $\psi(A)$, considered as column vectors, consists of all $\mathbb{F}_q[t]$-linear combinations of the columns of C. Thus $\psi(A) = C \cdot X$ when X is as in problem 10. Show that when $t^{-1}U$ is the open subset as in problem 10, then $C \cdot t^{-1}U$ is a fundamental domain for $\psi(A)$ in $Y = \oplus_{i=1}^n F_{w_i}$. Conclude from this and problems 10 and 11 that $\mu(Y/\psi(A)) = q^{-v_\infty(\det(C)) - n}$.
12. For C as in problem 11, show that $-v_\infty(\det(C)) = \deg(\mathrm{Disc}(A/\mathbb{F}_q[t]))/2$, where the degree of the disciminant ideal $\mathrm{Disc}(A/\mathbb{F}_q[t])$ of $\mathbb{F}_q[t]$ is defined to be the degree of a monic generator

of this ideal. The formula at the end of problem 11 then becomes

$$\mu(Y/\psi(A)) = q^{\deg(\mathrm{Disc}(A/\mathbb{F}_q[t]))/2-n}.$$

If you have seen the Hurwitz formula for covers of curves, try checking that the right hand side of this formula gives

$$\mu(Y/\psi(A)) = q^{(2g(F)-2)/2} \tag{3.1}$$

when $g(F)$ is the genus of the smooth projective curve over $\mathbb{F}_q$ with function field F. The constant $q^{2g(F)-2}$ is the function field counterpart of the absolute value of the disciminant of the ring of integers of a number field. The equation (3.1) is the counterpart of the corresponding formula for totally real number fields.

13. Suppose S is an open compact subgroup of $Y = \oplus_{i=1}^n F_{w_i}$. Let T be a finitely generated $\mathbb{F}_q[t]$-submodule of Y which is co-compact. Show that if $\mu(S) > \mu(Y/T)$, then there is a non-zero element $s \in S \cap T$. Thus S is a function field counterpart of the convex symmetric subsets which come up in the classical theory of geometry of numbers, and there is no power of 2 needed in the function field case.
14. Use problems 12 and 13 to show the following counterpart of the classical Minkowski bound for the ideal classgroup of A. Suppose $\mathcal{C}$ is a non-zero integral ideal of A. Prove that there is a non-zero element $x \in \mathcal{C}$ such that

$$[A : Ax] \le q \cdot q^{(2g(F)-2)/2}[A : \mathcal{C}] \tag{3.2}$$

Here the index $[A : \mathcal{C}]$ is the counterpart of the norm of an integral ideal of the ring of integers of a number field, $q^{(2g(F)-2)/2}$ is the counterpart of the square root of the discriminant of the field, and the Minkowski constant in the "totally real" function field case becomes simply q. Compare this result to what you can prove using Riemann-Roch.
Hints: Let w_1 be a fixed choice of a valuation of F over v_∞. Try using a compact open subset of $Y = \oplus_{i=1}^n F_{w_i}$ of the form

$$S = (t^c O_{w_1}) \times \prod_{i=2}^n O_{w_i}$$

for a well chosen integer c.