

# CUSPS OF MINIMAL NON-COMPACT ARITHMETIC HYPERBOLIC 3-ORBIFOLDS

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ABSTRACT. In this paper we count the number of cusps of minimal non-compact finite volume arithmetic hyperbolic 3-orbifolds. We show that for each  $N$ , the orbifolds of this kind which have exactly  $N$  cusps lie in a finite set of commensurability classes, but either an empty or an infinite number of isometry classes.

## 1. INTRODUCTION.

In this paper we count the number of cusps of minimal non-compact finite volume arithmetic hyperbolic 3-orbifolds. An orbifold of this kind is isometric to  $\mathbf{H}^3/\Gamma$ , where  $\mathbf{H}^3$  is hyperbolic upper half space and  $\Gamma$  is a maximal discrete arithmetic subgroup in  $\mathrm{PGL}_2(k)$  for some imaginary quadratic field  $k$ .

It is well known (cf. §3 below) that the cusps of the orbifold  $\mathbf{H}^3/\Gamma$  correspond to  $\Gamma$ -equivalence classes of points of  $\mathbb{P}_k^1$  under the action of  $\mathrm{PGL}_2(k)$  on  $\mathbb{P}_k^1$ . It was first noted by Bianchi [2] that  $\mathbf{H}^3/\mathrm{PSL}_2(O_k)$  has  $h_k$  cusps where  $h_k$  is the class number of  $k$ . By work of Allan [1] and Schmidt [7], there is a unique maximal arithmetic subgroup  $\Gamma_{\phi,\phi}$  of  $\mathrm{PGL}_2(k)$  which contains  $\mathrm{PSL}_2(O_k)$ . Let  $Cl(k)$  be the ideal class group of  $k$ , and let  $h_{k,2}$  be the order of  $Cl(k)/(2 \cdot Cl(k))$ . It follows from the work of Vinberg in [11, §2] that  $\mathbf{H}^3/\Gamma_{\phi,\phi}$  has  $h_k/h_{k,2}$  cusps. (Some closely related results are proved by Elstrodt, Grunewald and Mennicke in [4, §7.2,7.4]). In particular, since there are only finitely many imaginary quadratic number fields of a fixed class number, for any given  $N$  there are only finitely orbifolds  $\mathbf{H}^3/\Gamma_{\phi,\phi}$  as above which have  $N$  cusps.

The objective of this paper is to generalize the above result of Vinberg to an arbitrary maximal arithmetic subgroup  $\Gamma$  of  $\mathrm{PGL}_2(k)$ . To state the main theorem, recall that in [3], Borel described for each pair  $(S, S')$  of finite disjoint sets of finite places of  $k$  a discrete finite covolume subgroup  $\Gamma_{S,S'}$  of  $\mathrm{PGL}_2(k)$ . We recall the definition of  $\Gamma_{S,S'}$  in §2. Borel showed that each maximal finite covolume discrete subgroup of  $\mathrm{PGL}_2(k)$  is conjugate to  $\Gamma_{S,S'}$  for some  $(S, S')$ .

The main result of this paper is:

**Theorem 1.1.** *Let  $Cl(k)$  be the ideal class group of  $k$ . The number of cusps of  $\mathbf{H}/\Gamma_{S,S'}$  is*

$$2^n \frac{h_k}{h_{k,2}}$$

where  $h_k$  is the class number of  $k$ ,  $h_{k,2}$  is the order of  $Cl(k)/(2 \cdot Cl(k))$ ,  $0 \leq n \leq \#S$  and  $2^n$  is the order of the subgroup of  $Cl(k)/(2 \cdot Cl(k))$  generated by the classes of prime ideals determined by the places in  $S$ .

This Theorem and work of Siegel in [8] leads to a proof of the following Corollary.

**Corollary 1.2.** *Let  $N$  be a positive integer, and let  $C(N)$  be the set of isometry classes of minimal finite volume arithmetic hyperbolic three-orbifolds which have exactly  $N$  cusps.*

- a. *Only finitely many commensurability classes are represented by the elements of  $C(N)$ .*

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- b. If  $C(N)$  is not empty, there are infinitely many elements of  $C(N)$  which are commensurable to each element of  $C(N)$ .

The proof of part (a) of this Corollary is not effective, though it can be made effective up to at most one exceptional commensurability class using work of Tatuzawa in [9]. Finding an effective proof is equivalent to the problem of showing that there are only finitely many imaginary quadratic fields  $k$  such that  $h/h_{k,2}$  is bounded above by a given constant. Such a proof appears to be beyond present methods.

This paper is organized in the following way. In §2 we recall Borel's definition of  $\Gamma_{S,S'}$ . In §3 we recall some well-known facts concerning cusps of non-compact arithmetic three-orbifolds. In §4 and §5 we analyze the cusps of certain orbifolds defined by congruence subgroups of  $\Gamma_{S,S'}$ . This leads to the proof of Theorem 1.1 in §6 - §8. The main techniques used in §4 - §8 are Borel's work, the Strong Approximation Theorem for  $\mathrm{SL}_2$ , and an argument of Swan [10] for constructing matrices satisfying various congruence conditions which send a prescribed point of  $\mathbb{P}_k^1$  to another prescribed point. Corollary 1.2 is proved in §9.

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## 2. BOREL'S SUBGROUPS.

Let  $k$  be an imaginary quadratic field, with ring of integers  $O = O_k$ . Let  $S$  and  $S'$  be finite disjoint subsets of the set of all finite places  $v$  of  $k$ . For each such  $v$ , let  $k_v$  be the completion of  $k$  at  $v$ . Let  $\pi_v$  be a uniformizer in the ring of integers  $O_v$  of  $k_v$ . Define  $\mathcal{D}_v = \mathrm{Mat}_2(O_v)$ , and let  $\mathcal{D}'_v$  be the maximal  $O_v$ -order of all matrices of the form

$$(2.1) \quad M = \begin{pmatrix} a & \pi_v b \\ \pi_v^{-1} c & d \end{pmatrix}$$

in which  $a, b, c, d \in O_v$ . Define  $K_{1,v} = \mathrm{PGL}_2(O_v)$ , so that  $K_{1,v}$  is the image of  $\mathcal{D}_v^*$  in  $\mathrm{PGL}_2(k_v)$ . Let  $K'_{1,v}$  to be the image of  $\mathcal{D}'_v$  in  $\mathrm{PGL}_2(k_v)$ . Finally, let  $K_{2,v}$  be the group generated by  $K_{1,v} \cap K'_{1,v}$  together with image in  $\mathrm{PGL}_2(k_v)$  of the element

$$(2.2) \quad w_v = \begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}$$

Then  $K_{1,v}$  and  $K'_{1,v}$  are the stabilizers in  $\mathrm{PGL}_2(k_v)$  of adjacent vertices of the Bruhat-Tits building of  $\mathrm{SL}_2(k_v)$ , and  $K_{2,v}$  is the stabilizer the edge joining these vertices. In [3] Borel defines

$$(2.3) \quad \Gamma_{S,S'} = \{g \in \mathrm{PGL}_2(k) : g \in K_{2,v} \text{ (resp. } K'_{1,v}, \text{ resp. } K_{1,v}) \text{ if } v \in S \text{ (resp. } v \in S', v \notin S \cup S')\}$$

It is shown in [3, Prop. 4.4] that every maximal arithmetic discrete subgroup of  $\mathrm{PGL}_2(k)$  is conjugate to  $\Gamma_{S,S'}$  for some  $S$  and  $S'$ . Not all of the  $\Gamma_{S,S'}$  need be maximal (cf. [3, §4.4]). By [3, Prop. 4.10, Thm. 4.6], the groups  $\Gamma_{S,S'}$  for a fixed  $S$  lie in finitely many conjugacy classes inside  $\mathrm{PGL}_2(k)$ , while as  $S$  varies these groups lie in infinitely many distinct conjugacy classes.

## 3. CUSPS.

Suppose  $\Gamma$  is any discrete arithmetic subgroup of  $\mathrm{PGL}_2(k)$  having finite covolume. An element  $\sigma \in \Gamma$  is parabolic if it fixes a unique point of  $\mathbb{P}_{\mathbb{C}}^1$ , and such a fixed point is called a cusp of  $\Gamma$  (compare [6, p. 7-8]). The cusps of the orbifold  $\mathbf{H}^3/\Gamma$  are the  $\Gamma$ -equivalence classes of cusps of  $\Gamma$ .

**Lemma 3.1.** *The cusps of  $\Gamma$  are the points in  $\mathbb{P}_k^1 = k \cup \{\infty\}$ , so that the cusps of  $\mathbf{H}^3/\Gamma$  are the orbits of  $\Gamma$  acting on  $\mathbb{P}_k^1$ .*

**Proof:** We first show that  $\Gamma$  has the same cusps as any group  $\Gamma'$  commensurable to  $\Gamma$ . For this, it will suffice to consider the case in which  $\Gamma'$  has finite index in  $\Gamma$ . Clearly the cusps of  $\Gamma'$  are cusps for  $\Gamma$ . Conversely, suppose  $z$  is a cusp of  $\Gamma$ , so  $z$  is the only point of  $\mathbb{P}_{\mathbb{C}}^1$  fixed by a parabolic element  $\sigma \in \Gamma$ . Then  $\sigma^n$  is a parabolic element of  $\Gamma'$  fixing  $z$  when  $n = [\Gamma : \Gamma']$ , so  $z$  is also a cusp of  $\Gamma'$ . We can thus reduce to the case in which  $\Gamma = \Gamma_{S,S'}$  for some  $S$  and  $S'$ .

Suppose  $z$  is a cusp of  $\Gamma_{S,S'} \subset \mathrm{PGL}_2(k)$ . Since  $z$  is the only fixed point of some  $M \in \mathrm{GL}_2(k)$  acting on  $\mathbb{P}_{\mathbb{C}}^1$ , the quadratic formula implies that  $z$  must lie in  $\mathbb{P}_k^1$ . Thus we now must show each  $z \in \mathbb{P}_k^1$  is a cusp.

If  $b$  is a sufficiently divisible non-zero element of  $O$ , the matrix

$$(3.1) \quad M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

defines a parabolic element of  $\Gamma_{S,S'}$  which fixes  $\infty$ , so  $\infty$  is a cusp of  $\Gamma_{S,S'}$ . Suppose now that  $z \in k \subset \mathbb{P}_k^1$ . There is then a matrix  $T$  in  $\mathrm{GL}_2(k)$  such that  $T \cdot \infty = z$ . This implies  $z$  is a cusp of the discrete group  $T\Gamma_{S,S'}T^{-1}$ , since  $TMT^{-1}$  defines a parabolic element of this group fixing  $z$ . However,  $T\Gamma_{S,S'}T^{-1}$  and  $\Gamma_{S,S'}$  are commensurable, so they have the same cusps, which proves the Lemma.

In the following sections we analyze equivalence classes of cusps under the action of various subgroups  $\Gamma$  of  $\Gamma_{S,S'}$ .

#### 4. THE PRINCIPAL CONGRUENCE SUBGROUP OF $\Gamma_{S,S'}$ .

We consider in this section the following subgroup of  $\Gamma_{S,S'}$ .

**Definition 4.1.** Let  $I$  be the two-by-two identity matrix. Define  $\Gamma(S, S')$  to be the subgroup of elements of  $\Gamma_{S,S'} \subset \mathrm{PGL}_2(k)$  which are the images of matrices  $M \in \mathrm{GL}_2(k)$  such that  $M - I \in \pi_v \mathrm{Mat}_2(O_v)$  for  $v \in S$ ,  $M \in \mathcal{D}'_v^*$  for  $v \in S'$  and  $M \in \mathrm{GL}_2(O_v)$  for  $v \notin S \cup S'$ .

We will first describe the  $\Gamma(S, S')$ -equivalent cusps of  $\Gamma(S, S')$ . By Lemma 3.1, this is the same as describing the cusps of  $\mathbf{H}^3/\Gamma(S, S')$ , and the orbits of  $\Gamma(S, S')$  acting on  $\mathbb{P}_k^1$ .

Define  $\mathcal{I}(k)$  to be the multiplicative group of fractional ideals of  $k$ . For  $v$  a finite place of  $k$ , let  $\mathcal{P}(v)$  be the prime ideal of  $O$  determined by  $v$ . If  $T$  is a finite set of finite places of  $k$ , define  $\mathcal{P}(T) = \prod_{v \in T} \mathcal{P}(v)$ . Define  $L'(S)$  to be the set of triples  $(J, \alpha_0, \alpha_1)$  in which  $J \in \mathcal{I}(k)$  and  $\alpha_0$  and  $\alpha_1$  are generators of  $J/(\mathcal{P}(S) \cdot J)$  as a finite  $O$ -module. An element  $\lambda \in k^*$  acts on  $L'(S)$  by sending  $(J, \alpha_0, \alpha_1)$  to  $(\lambda \cdot J, \lambda \cdot \alpha_0, \lambda \cdot \alpha_1)$ . Define  $L(S) = L'(S)/k^*$  to be the set of orbits in  $L'(S)$  under this action of  $k^*$ .

**Definition 4.2.** Define a map  $\Psi : \mathbb{P}_k^1 \rightarrow L(S)$  in the following way. Fix an element  $t(S, S') \in \mathcal{P}(S')$  such that the ideal  $t(S, S')O$  equals  $\mathcal{P}(S') \cdot \mathcal{A}$  for some ideal  $\mathcal{A}$  prime to  $\mathcal{P}(S \cup S')$ . Suppose  $(x_0 : x_1)$  are homogeneous coordinates for a point of  $\mathbb{P}_k^1$ . Define  $J$  to be the fractional  $O$ -ideal  $O \cdot x_0 + \mathcal{P}(S') \cdot x_1$  of  $k$ . Let  $\beta_0 = x_0$  and  $\beta_1 = t(S, S')x_1$ , so that  $\beta_0$  and  $\beta_1$  are elements of  $J$ . Define  $\alpha_i$  to be the image of  $\beta_i$  in  $J/(\mathcal{P}(S) \cdot J)$  for  $i = 0, 1$ . Define

$$(4.1) \quad \Psi((x_0 : x_1)) = [(J, \alpha_0, \alpha_1)]$$

to be the class of  $(J, \alpha_0, \alpha_1)$  in  $L(S)$ . The other homogeneous coordinates for  $(x_0 : x_1)$  have the form  $(\lambda \cdot x_0 : \lambda \cdot x_1)$  for some  $\lambda \in k^*$ , so  $\Psi$  is well-defined.

**Proposition 4.3.** *The map  $\Psi$  is surjective, and its fibers are exactly the  $\Gamma(S, S')$ -equivalent cusps of  $\Gamma(S, S')$ .*

**Proof:** Let us first check surjectivity. Suppose  $(J, \alpha_0, \alpha_1) \in L'(S)$ . We first claim that there is an  $x_1 \in k^*$  such that  $\mathcal{P}(S') \cdot x_1 \subset J$  and  $t(S, S')x_1 \in J$  has class  $\alpha_1$  in  $J/(\mathcal{P}(S) \cdot J)$ . Such an  $x_1$  exists because we can find an  $x_1 \in \mathcal{P}(S')^{-1}J$  satisfying the appropriate congruence conditions at the places in  $S$  because  $S$  and  $S'$  are disjoint. Choose  $x_0 \in J$  to have class  $\alpha_0$  in  $J/(\mathcal{P}(S) \cdot J)$ , and so that  $O_v \cdot x_0 = O_v \cdot J$  for the finitely many finite places  $v$  of  $k$  which are not in  $S$  where  $O_v \cdot \mathcal{P}(S')x_1$  is not equal to  $O_v \cdot J$ . We can find such an  $x_0$  since these conditions amount to congruence conditions at a finite set of finite places of  $k$ . We show  $\Psi((x_0 : x_1))$  is the class of  $(J, \alpha_0, \alpha_1)$  in  $L(S)$ . By construction,  $x_0$  has class  $\alpha_0$  in  $J/\mathcal{P}(S)J$ , while  $t(S, S')x_1$  has class  $\alpha_1$  in  $J/(\mathcal{P}(S) \cdot J)$ . Hence we only have to check that  $J' = O \cdot x_0 + \mathcal{P}(S')x_1$  is equal to  $J$ . Clearly  $J' \subset J$ . Since  $\alpha_0 \equiv x_0$  and  $\alpha_1 \equiv t(S, S')x_1$  together generate  $J/(\mathcal{P}(S) \cdot J)$  as an  $O$ -module, we have  $O_v \cdot J' = O_v \cdot J$  if  $v \in S$ .

However, for  $v \notin S$ , we chose  $x_0$  so that  $O_v \cdot x_0 = O_v \cdot J$  if  $O_v \cdot \mathcal{P}(S')x_1$  is not equal to  $O_v \cdot J$ . Thus  $O_v \cdot J' = O_v \cdot J$  for all such  $v$ , and we conclude  $J' = J$ .

We now consider the fibers of  $\Psi$ . Suppose  $(x_0 : x_1)$  and  $(x'_0 : x'_1)$  are two points having the same image under  $\Psi$ . After multiplying  $x'_0$  and  $x'_1$  by a suitable  $\lambda \in k^*$ , we can assume the following is true:

$$(4.2) \quad J = O x_0 + \mathcal{P}(S')x_1 = O x'_0 + \mathcal{P}(S')x'_1$$

$$(4.3) \quad x_0 \equiv x'_0 \equiv \alpha_0 \pmod{\mathcal{P}(S)J}$$

and

$$(4.4) \quad t(S, S')x_1 \equiv t(S, S')x'_1 \equiv \alpha_1 \pmod{\mathcal{P}(S)J}.$$

We wish to show that there is a matrix  $M \in \mathrm{GL}_2(k)$  such that

$$(4.5) \quad M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix}$$

and

$$(4.6) \quad M - I \in \pi_v \mathrm{Mat}_2(O_v) \quad \text{for } v \in S,$$

$$(4.7) \quad M \in \mathcal{D}'_v{}^* \quad \text{for } v \in S',$$

$$(4.8) \quad M \in \mathrm{GL}_2(O_v) \quad \text{for } v \notin S \cup S'.$$

We adapt an argument of Swan in [10, Prop. 3.10] to construct  $M$ . There are two exact sequences of  $O$ -modules

$$(4.9) \quad 0 \longrightarrow \mathcal{B} \longrightarrow O \oplus \mathcal{P}(S') \xrightarrow{l'} J \longrightarrow 0$$

$$(4.10) \quad 0 \longrightarrow \mathcal{C} \longrightarrow O \oplus \mathcal{P}(S') \xrightarrow{l} J \longrightarrow 0$$

in which  $l$  and  $l'$  are defined for  $(a, b) \in O \oplus \mathcal{P}(S')$  by

$$(4.11) \quad l(a, b) = ax_0 + bx_1 \quad \text{and} \quad l'(a, b) = ax'_0 + bx'_1.$$

Since  $J$  is a projective  $O$ -module, these sequences split, giving isomorphisms

$$(4.12) \quad O \oplus \mathcal{P}(S') = J \oplus \mathcal{B} \quad \text{and} \quad O \oplus \mathcal{P}(S') = J \oplus \mathcal{C}.$$

Again using the fact that  $O$  is a Dedekind ring, these isomorphisms imply that there is an isomorphism  $\phi : \mathcal{B} \rightarrow \mathcal{C}$  of projective rank one  $O$ -modules.

Let  $s$  be a unit of  $O$ , and suppose  $W \in \mathrm{Hom}_O(J, \mathcal{C})$ . We define an  $O$ -linear map

$$(4.13) \quad \theta_{s,W} : O \oplus \mathcal{P}(S') = J \oplus \mathcal{B} \rightarrow J \oplus \mathcal{C} = O \oplus \mathcal{P}(S')$$

by

$$(4.14) \quad \theta_{s,W}(j \oplus a) = j \oplus (s\phi(a) + W(j))$$

for  $j \in J$  and  $a \in \mathcal{B}$ . Then  $\theta_{s,W}$  fits into a commutative diagram

$$(4.15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{B} & \longrightarrow & O \oplus \mathcal{P}(S') & \xrightarrow{l'} & J & \longrightarrow & 0 \\ & & s\phi \downarrow & & \downarrow \theta_{s,W} & & \downarrow 1 & & \\ 0 & \longrightarrow & \mathcal{C} & \longrightarrow & O \oplus \mathcal{P}(S') & \xrightarrow{l} & J & \longrightarrow & 0 \end{array}$$

Since  $s\phi$  is an isomorphism, and  $1 : J \rightarrow J$  is the identity map, we conclude that  $\theta_{s,W}$  is an automorphism. Furthermore,  $\det_O(\theta_{s,W}) = s \cdot \det_O(\theta_{1,W}) = s \cdot \det_O(\theta_{1,0})$  is independent of the choice of  $W$ , so we can choose  $s$  (depending on  $\phi$ ) so that  $\det(\theta_{s,W}) = 1$  for all  $W$ .

Define

$$(4.16) \quad M_{s,W} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

to be the matrix of  $\theta_{s,W}$  when we view elements of  $O \oplus \mathcal{P}(S') \subset k \oplus k$  as column vectors. Here

$$(4.17) \quad \begin{aligned} \alpha \in \text{Hom}_O(O, O) &= O, \\ \beta \in \text{Hom}_O(\mathcal{P}(S'), O) &= \mathcal{P}(S')^{-1}, \\ \gamma \in \text{Hom}_O(O, \mathcal{P}(S')) &= \mathcal{P}(S') \\ \delta \in \text{Hom}_O(\mathcal{P}(S'), \mathcal{P}(S')) &= O. \end{aligned}$$

Thus the transpose  $M_{s,W}^{tr}$  of  $M_{s,W}$  is an element of  $\text{SL}_2(k)$  satisfying conditions (4.7) and (4.8), while  $M_{s,W}^{tr} \in \text{SL}_2(O_v)$  for  $v \in S$ . The commutativity of (4.15) shows

$$(4.18) \quad x'_0 = l' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = l \left( M_{s,W} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = l \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \alpha x_0 + \gamma x_1$$

and

$$(4.19) \quad t(S, S')x'_1 = l \left( M_{s,W} \cdot \begin{pmatrix} 0 \\ t(S, S') \end{pmatrix} \right) = l \begin{pmatrix} \beta \cdot t(S, S') \\ \delta \cdot t(S, S') \end{pmatrix} = \beta \cdot t(S, S') \cdot x_0 + \delta \cdot t(S, S') \cdot x_1$$

This gives the matrix equation

$$(4.20) \quad M_{s,W}^{tr} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix}$$

We now show that we can choose  $W \in \text{Hom}_O(J, \mathcal{C})$  so that  $M_{s,W}^{tr} = M$  will satisfy (4.6), i.e. so that  $M - I \in \pi_v \text{Mat}_2(O_v)$  for  $v \in S$ . This will complete the proof that cusps having the same image under  $\Psi$  are  $\Gamma(S, S')$ -equivalent.

For  $v \in S$ , let  $k(v) = O/\mathcal{P}(v)$  and let  $J_v$  be the localization of  $J$  at  $v$ . Since  $t(S, S') \in O_v^*$  and  $t(S, S')x_1 \in J$ , we have  $x_1 \in J_v$  for  $v \in S$ . Define  $\beta_{i,v}$  be the image of  $x_i$  in the one-dimensional  $k(v)$ -vector space  $J(v) = J_v/\mathcal{P}(v)J_v$ . From (4.20), (4.3) and (4.4) we know that for  $v \in S$ ,  $M_{s,W}^{tr} \in \text{SL}_2(O_v)$  fixes the vector  $\beta(v) = (\beta_{0,v}, \beta_{1,v})$  in  $J(v) \oplus J(v)$ . This  $\beta(v)$  is not the zero vector, since  $\alpha_0$  and  $\alpha_1$  together generate  $J/\mathcal{P}(S)J$  and  $\alpha_0 = x_0 \bmod \mathcal{P}(S)J$  and  $\alpha_1 = t(S, S')x_1 \bmod \mathcal{P}(S)J$ . Thus the image  $M_{s,W,v}^{tr}$  of  $M_{s,W}^{tr}$  in  $\text{SL}_2(k(v))$  lies in the stabilizer of  $\beta(v)$ , and this stabilizer has order  $\#k(v)$  since  $\beta(v)$  is non-zero. Letting  $v$  range over  $S$ , we see that the image of  $M_{s,W}^{tr}$  in  $T = \prod_{v \in S} \text{SL}_2(k(v))$  lies in a subgroup of matrices which has order  $N = \prod_{v \in S} \#k(v)$ . However, as  $W$  ranges over  $\text{Hom}(J, \mathcal{C})$ , the image of  $M_{s,W}^{tr}$  in  $T$  also ranges over a set of  $N$  matrices, since each of  $J$  and  $\mathcal{C}$  are rank one projective  $O$ -modules. It follows that we can choose  $W$  so that  $M_{s,W}^{tr}$  has image the identity element of  $T$ , as required.

The last statement we have to prove is that  $\Gamma(S, S')$ -equivalent cusps have the same image under  $\Psi$ . Suppose  $M = M_{s,W}^{tr}$  satisfies (4.20) and has the properties described in Definition (4.1). It will suffice to show (4.2), (4.3) and (4.4) hold. For (4.2), observe that the containments in (4.17) show

$$(4.21) \quad Ox'_0 + \mathcal{P}(S')x'_1 = O(\alpha \cdot x_0 + \gamma \cdot x_1) + \mathcal{P}(S')(\beta \cdot x_0 + \delta \cdot x_1) \subset Ox_0 + \mathcal{P}(S')x_1$$

Since  $M^{-1}$  also satisfies the conditions in (4.1) and takes the cusp  $(x'_0 : x'_1)$  back to  $(x_0 : x_1)$ , we can interchange  $(x'_0 : x'_1)$  and  $(x_0 : x_1)$  to conclude that (4.2) holds. The proof of (4.3) and (4.4) is similar using the properties of  $M$  in Definition (4.1).

## 5. THE BOREL CONGRUENCE SUBGROUP OF $\Gamma_{S,S'}$ .

We consider in this section the following subgroup of  $\Gamma_{S,S'}$ .

**Definition 5.1.** For  $v$  a finite place of  $k$ , let  $B_v \subset \text{GL}_2(O_v)$  be the subgroup of invertible matrices of the form

$$(5.1) \quad \begin{pmatrix} a & \pi_v b \\ c & d \end{pmatrix}$$

in which  $a, b, c, d \in O_v$ . Define  $\Gamma_0(S, S')$  to be the subgroup of elements of  $\Gamma_{S,S'} \subset \text{PGL}_2(k)$  which are the images of matrices  $M \in \text{GL}_2(k)$  such that that  $M \in B_v$  for  $v \in S$ ,  $M \in \mathcal{D}'_v^*$  for  $v \in S'$  and  $M \in \text{GL}_2(O_v)$  for  $v \notin S \cup S'$ .

Note that the image of  $B_v$  in  $\mathrm{PGL}_2(k_v)$  is the group  $K_{1,v} \cap K'_{1,v}$  defined in §2. Thus  $\Gamma_0(S, S') \subset \Gamma_{S, S'}$ , while  $\Gamma(S, S') \subset \Gamma_0(S, S')$ .

**Definition 5.2.** Define  $L_0(S)$  to be the set of pairs  $([J], \{\beta_v\}_{v \in S})$  in which  $[J]$  is an element of the ideal class group of  $k$ , and for each  $v \in S$ ,  $\beta_v$  is either 0 or 1. Define  $r : L(S) \rightarrow L_0(S)$  to be the map which sends a triple  $(J, \alpha_0, \alpha_1) \in L'(S)$  representing an element of  $L(S)$  to  $([J], \{\beta_v\}_{v \in S})$ , where  $[J]$  is the ideal class of  $J \in I(k)$ , and  $\beta_v = 0$  (resp. 1) if  $\alpha_0 \equiv 0 \pmod{\pi_v J}$  (resp. if  $\alpha_0 \not\equiv 0 \pmod{\pi_v J}$ ).

**Proposition 5.3.** *Let  $\Psi : \mathbb{P}_k^1 \rightarrow L(S)$  be the map of Proposition 4.3. The composition  $r \circ \Psi : \mathbb{P}_k^1 \rightarrow L_0(S)$  is surjective, and the fibers of this map are the  $\Gamma_0(S, S')$ -equivalent cusps of  $\Gamma_0(S, S')$ .*

**Proof:** Recall that  $L'(S)$  consists of the triples  $(J, \alpha_0, \alpha_1)$  in which  $J \in I(k)$  and  $\alpha_0$  and  $\alpha_1$  are generators of  $J/\mathcal{P}(S)J$ . Since  $\mathcal{P}(S) = \prod_{v \in S} \mathcal{P}(v)$ , we see that we can choose  $\alpha_0$  and  $\alpha_1$  to have prescribed classes  $\alpha_0(v), \alpha_1(v) \in J/\mathcal{P}(v)$  as  $v$  ranges over  $S$  provided that for no such  $v$  are both  $\alpha_0(v)$  and  $\alpha_1(v)$  trivial. This implies  $r$  is surjective, so  $r \circ \Psi$  is surjective by Proposition 4.3.

Consider now the action of a matrix  $M \in \mathrm{GL}_2(k)$  satisfying the hypotheses of Definition 5.1 on  $\Psi(x_0 : x_1) = [(J, \alpha_0, \alpha_1)] \in L(S)$ , where  $(J, \alpha_0, \alpha_1) \in L'(S)$  is as in Definition 4.2 and  $[(J, \alpha_0, \alpha_1)]$  is the class of  $(J, \alpha_0, \alpha_1)$  in  $L(S)$ . From  $J = O \cdot x_0 + \mathcal{P}(S') \cdot x_1$  and the hypotheses on  $M$  we see that  $J = O \cdot x'_0 + \mathcal{P}(S') \cdot x'_1$  when

$$(5.2) \quad \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Recall that  $\alpha_0$  (resp.  $\alpha_1$ ) is the image in  $J/\mathcal{P}(S)$  of  $x_0$  (resp.  $t(S, S')x_1$ ). Suppose

$$(5.3) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Define

$$(5.4) \quad M' = \begin{pmatrix} a & b \cdot t(S, S')^{-1} \\ t(S, S')c & d \end{pmatrix}$$

We find from (5.2) that  $\Psi(x'_0 : x'_1) = (J, \alpha'_0, \alpha'_1)$ , where  $\alpha'_0$  and  $\alpha'_1$  are elements of  $J/\mathcal{P}(S)$  given by the following residue classes  $\alpha'_0(v), \alpha'_1(v) \in J_v/\mathcal{P}(v)J_v = J/\mathcal{P}(v)J$  for  $v \in S$ :

$$(5.5) \quad \begin{pmatrix} \alpha'_0(v) \\ \alpha'_1(v) \end{pmatrix} = M' \cdot \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \end{pmatrix}.$$

The number  $t(S, S') \in k$  is a unit at each  $v \in S$ , so  $M' \in B_v$  for such  $v$  because  $M \in B_v$ . Thus

$$(5.6) \quad a, d \in O_v^*, \quad bt(S, S')^{-1} \in \pi_v O_v \quad \text{and} \quad t(S, S')c \in O_v$$

This implies  $\alpha'_0(v) = 0$  if and only if  $\alpha_0(v) = 0$ . It follows that  $r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1)$ , so  $\Gamma_0(S, S')$ -equivalent cusps have the same image under  $r \circ \Psi$ .

To complete the proof of Proposition 5.3, we have to show that two points  $(x_0 : x_1)$  and  $(x'_0 : x'_1)$  with the same image under  $r \circ \Psi$  are  $\Gamma_0(S, S')$ -equivalent. After multiplying  $x'_0$  and  $x'_1$  by a suitable scalar, we can assume

$$(5.7) \quad J = O \cdot x_0 + \mathcal{P}(S') \cdot x_1 = O \cdot x'_0 + \mathcal{P}(S') \cdot x'_1$$

Furthermore, on defining  $\alpha_0(v)$ ,  $\alpha_1(v)$ ,  $\alpha'_0(v)$  and  $\alpha'_1(v)$  to be the images of  $x_0$ ,  $t(S, S')x_1$ ,  $x'_0$  and  $t(S, S')x'_1$  in  $J/\mathcal{P}(v)J$ , we see that  $\alpha_0(v) = 0$  if and only if  $\alpha'_0(v) = 0$  for  $v \in S$ , since  $r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1)$ . Furthermore,  $\alpha_1(v) \neq 0$  if  $\alpha_0(v) = 0$ , and similarly  $\alpha'_1(v) \neq 0$  if  $\alpha'_0(v) = 0$ . This implies there is a lower triangular matrix  $m_v \in \mathrm{SL}_2(O_v/\pi_v O_v)$  such that

$$(5.8) \quad \begin{pmatrix} \alpha'_0(v) \\ \alpha'_1(v) \end{pmatrix} = m_v \cdot \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \end{pmatrix}$$

We now use the Strong Approximation Theorem for  $\mathrm{SL}_2$  to conclude that there is  $M \in \mathrm{SL}_2(k)$  which satisfies the hypotheses of Definition 5.1 such that when we write  $M$  in the form (5.3) and let

$M'$  be as in (5.4), then  $M' \in \mathrm{SL}_2(O_v)$  for  $v \in S$  satisfies the congruence  $M' \equiv m_v \pmod{\pi_v \mathrm{Mat}_2(O_v)}$ . We conclude from this that

$$\Psi(M \cdot (x_0 : x_1)) = \Psi(x'_0 : x'_1)$$

so that  $M \cdot (x_0 : x_1)$  and  $(x'_0 : x'_1)$  are  $\Gamma(S, S')$ -equivalent cusps by Proposition 4.3. Since  $\Gamma(S, S') \subset \Gamma_0(S, S')$  and  $M \cdot (x_0 : x_1)$  is  $\Gamma_0(S, S')$  equivalent to  $(x_0 : x_1)$  by our construction of  $M$ , this proves  $(x_0 : x_1)$  and  $(x'_0 : x'_1)$  are  $\Gamma_0(S, S')$ -equivalent cusps.

**Corollary 5.4.** *The number of  $\Gamma_0(S, S')$ -equivalence classes of cusps of  $\Gamma_0(S, S')$  is  $2^{\#S} h_k$ , where  $h_k$  is the class number of  $k$ .*

## 6. $\Gamma_{S, S'}$ -INEQUIVALENT CUSPS.

In this section we will prove Theorem 1.1. The proof is based on the following two results, which will be proved in §7 and §8, respectively.

**Proposition 6.1.** *Let  $C_0(S, S')$  be the set of  $\Gamma_0(S, S')$ -equivalence classes of points of  $P_k^1$ . Since  $\Gamma_0(S, S') \subset \Gamma_{S, S'}$ , the group  $\Gamma_{S, S'}$  acts on  $C_0(S, S')$ . Each  $\Gamma_{S, S'}$ -orbit in  $C_0(S, S')$  has  $[\Gamma_{S, S'} : \Gamma_0(S, S')]$  elements.*

**Proposition 6.2.** *Define  $h_{k,2}$  to be the order of  $Cl(k)/(2Cl(k))$  where  $Cl(k)$  is the class group of  $k$ . Define  $2^n$  to be the order of the subgroup of  $Cl(k)/(2 \cdot Cl(k))$  generated by the classes of prime ideals determined by the places in  $S$ . Then  $0 \leq n \leq \#S$  and*

$$(6.1) \quad [\Gamma_{S, S'} : \Gamma_0(S, S')] = 2^{\#S-n} h_{k,2}.$$

Theorem 1.1 is a consequence of these results in the following way. By Lemma 3.1 the set of  $\Gamma_{S, S'}$ -orbits in  $C_0(S, S')$  is the set of  $\Gamma_{S, S'}$ -equivalence classes of cusps of  $\Gamma_{S, S'}$ . Corollary 5.4 together with Propositions 6.1 and 6.2 show this number is

$$(6.2) \quad \frac{2^{\#S} h_k}{2^{\#S-n} h_{k,2}} = 2^n \frac{h_k}{h_{k,2}}$$

as stated in Theorem 1.1.

## 7. PROOF OF PROPOSITION 6.1.

We will need several Lemmas.

**Lemma 7.1.** *To prove Proposition 6.1, it will suffice to show the following. Suppose*

$$(7.1) \quad \sigma \in \Gamma_{S, S'}, \quad (x_0 : x_1) \in P_k^1, \quad (x'_0 : x'_1) = \sigma \cdot (x_0 : x_1) \quad \text{and} \quad r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1).$$

*Then  $\sigma$  lies in  $\Gamma_0(S, S')$ .*

**Proof:** This is clear from Proposition 5.3, which showed that the map  $r \circ \Psi : P_k^1 \rightarrow L_0(S)$  has fibers equal to the elements of  $C_0(S, S')$ .

We will assume from now on that hypothesis (7.1) holds.

**Definition 7.2.** Let  $(J, \alpha_0, \alpha_1)$  be the triple associated in Definition 4.2 to the ordered pair  $(x_0, x_1)$  of elements of  $k$  which are not both 0. Thus  $J = Ox_0 + \mathcal{P}(S')x_1$ , and  $\alpha_0$  and  $\alpha_1$  are the classes of  $x_0$  and  $t(S, S')x_1$  in  $J/\mathcal{P}(S)J$ . The class  $[(J, \alpha_0, \alpha_1)]$  of  $(J, \alpha_0, \alpha_1)$  in  $L(S)$  is equal to  $\Psi(x_0 : x_1)$ . Write

$$\begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

for some matrix  $M \in \mathrm{GL}_2(k)$  with image  $\sigma \in \Gamma_{S, S'}$  in  $\mathrm{PGL}_2(k)$ . Let  $(J', \alpha'_0, \alpha'_1)$  be the triple associated to  $(x'_0, x'_1)$ .

**Lemma 7.3.** *The element  $\sigma$  must be even at each  $v \in S$ , in the sense that  $\det(M)$  has even valuation at each  $v \in S$ .*

**Proof:** Suppose to the contrary that  $\sigma$  is odd at some place  $v \in S$ . From the definition of  $\Gamma_{S,S'}$  in §2, this implies that

$$(7.2) \quad M = \lambda_v \cdot w_v \cdot M_v$$

where  $\lambda_v \in k_v^*$ ,  $w_v$  is the matrix

$$(7.3) \quad w_v = \begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}$$

and

$$(7.4) \quad M_v = \begin{pmatrix} a & \pi_v b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v)$$

for some  $a, b, c, d \in O_v$ . Consider the localization  $J_v$  of  $J$  at  $v$ . Since  $\mathcal{P}(S')$  is prime to  $\mathcal{P}(v)$ , we have

$$(7.5) \quad J_v = O_v x_0 + O_v x_1 \subset k_v \quad \text{and} \quad J'_v = O_v x'_0 + O_v x'_1.$$

Since

$$(7.6) \quad \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \lambda_v \cdot w_v \cdot M_v \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

we see from (7.3) and (7.4) that

$$(7.7) \quad x'_0 = \lambda_v \cdot \pi_v \cdot (cx_0 + dx_1) \quad \text{and} \quad x'_1 = \lambda_v \cdot (ax_0 + \pi_v bx_1).$$

Here  $a, d \in O_v^*$ , since  $M_v$  in (7.4) is in  $\mathrm{GL}_2(O_v)$ . We claim

$$(7.8) \quad J'_v = \lambda_v \cdot (\pi_v O_v x_1 + O_v x_0).$$

To show this, let  $\mathrm{ord}_v : k_v \rightarrow \mathbf{Z} \cup \{\infty\}$  be the discrete valuation at  $v$ , normalized so that  $\mathrm{ord}_v(\pi_v) = 1$ . From (7.7) and (7.5) we have

$$J'_v = O_v x'_0 + O_v x'_1 \subset \lambda_v \cdot (\pi_v O_v x_1 + O_v x_0)$$

since  $a, b, c, d \in O_v$ . This containment must be an equality since (7.2) shows  $\mathrm{ord}_v(\det(M)) = \mathrm{ord}_v(\lambda_v^2) + 1$ , and this integer is the power of  $\#O_v/\pi_v O_v$  appearing in the generalized index

$$[O_v x_0 + O_v x_1 : \lambda_v \cdot (\pi_v O_v x_1 + O_v x_0)].$$

The first case we now must consider is when  $\mathrm{ord}_v(x_0) \leq \mathrm{ord}_v(x_1)$ . In this case, (7.5) and (7.8) show

$$(7.9) \quad J_v = O_v x_0 \quad \text{and} \quad J'_v = \lambda_v \cdot O_v x_0$$

Thus  $x_0 \not\equiv 0 \pmod{\pi_v J_v}$ , while (7.7) shows  $x'_0 \equiv 0$  in  $J'_v/\pi_v J'_v$ . This proves  $\alpha_0(v) \neq 0$  but  $\alpha'_0(v) = 0$ . In view of the description of the map  $r : L(S) \rightarrow L_0(S)$  in Definition 5.2, this forces  $r([(J, \alpha_0, \alpha_1)]) = r \circ \Psi(x_0 : x_1)$  to be different from  $r([(J', \alpha'_0, \alpha'_1)]) = r \circ \Psi(x'_0 : x'_1)$ . This contradicts hypothesis (7.1), so we conclude that this hypothesis forces  $\mathrm{ord}_v(x_0) > \mathrm{ord}_v(x_1)$ . In this case (7.5) and (7.8) imply

$$(7.10) \quad J_v = O_v x_1 \quad \text{and} \quad J'_v = \lambda_v \cdot \pi_v \cdot O_v x_1.$$

Since  $\mathrm{ord}_v(x_0) > \mathrm{ord}_v(x_1)$ , we find that  $x_0 \equiv 0 \pmod{\pi_v J_v}$ , while (7.7) implies  $x'_0 \not\equiv 0 \pmod{\pi_v J'_v}$ . Thus we get  $\alpha_0(v) = 0$  but  $\alpha'_0(v) \neq 0$ , again contradicting hypothesis (7.1). This contradiction proves Lemma 7.3.

**Corollary 7.4.** *The element  $\sigma \in \Gamma_{S,S'}$  is represented by a matrix  $M \in \mathrm{GL}_2(k)$  having the following properties. For each finite place  $v$  of  $k$ , there is an element  $x_v \in k_v^*$  together with elements  $a = a_v$ ,  $b = b_v$ ,  $c = c_v$  and  $d = d_v$  of  $O_v$  such that*

$$(7.11) \quad M = x_v \cdot M_v \quad \text{and} \quad \det(M_v) \in O_v^*,$$

$$(7.12) \quad x_v \in O_v^* \quad \text{for all but finitely many places } v$$



$$(7.13) \quad M_v = \begin{pmatrix} a & \pi_v b \\ c & d \end{pmatrix} \quad \text{if } v \in S$$

$$(7.14) \quad M_v = \begin{pmatrix} a & \pi_v b \\ \pi_v^{-1} c & d \end{pmatrix} \quad \text{if } v \in S'$$

$$(7.15) \quad M_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{if } v \notin S \cup S'$$

**Proof:** By [3, Prop. 4.4(iii)], if  $v$  is a finite place of  $k$  such that  $\Gamma_{S,S'}$  contains an element which is odd at  $v$ , then  $v \in S$ . We proved in Lemma 7.3 that  $\sigma$  must be even at each  $v \in S$ . Hence for each finite place  $v$ , there is an element  $x_v \in k_v^*$  such that  $2 \cdot \text{ord}_v(x_v) = \text{ord}_v(\det(M))$ . On defining  $M_v = x_v^{-1} \cdot M$ , it now follows from the definition of  $\Gamma_{S,S'}$  in (2.3) that  $M_v$  has properties (7.11) - (7.15).

**Corollary 7.5.** *With the notation of Corollary 7.4, let  $\mathcal{B} = \prod_v \mathcal{P}(v)^{\text{ord}_v(x_v)}$ . Then with the notation of (7.1), we have*

$$(7.16) \quad J' = Ox'_0 + \mathcal{P}(S')x'_1 = \mathcal{B} \cdot J = \mathcal{B} \cdot (Ox_0 + \mathcal{P}(S)x_1)$$

as fractional  $k$ -ideals.

**Proof:** Define

$$(7.17) \quad \begin{pmatrix} x_{v,0} \\ x_{v,1} \end{pmatrix} = M_v \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

for all  $v$ . Since  $M = x_v \cdot M_v$ , the localization  $J'_v$  at  $v$  satisfies

$$(7.18) \quad J'_v = (O_v x'_0 + \mathcal{P}(S')_v x'_1) = x_v \cdot (O_v x_{v,0} + O_v \mathcal{P}(S')_v x_{v,1})$$

However, the fact that  $\det(M_v) \in O_v^*$  together with the form of  $M_v$  in (7.13) - (7.15) ensures that

$$(7.19) \quad O_v x_{v,0} + O_v \mathcal{P}(S')_v x_{v,1} = O_v x'_0 + \mathcal{P}(S')_v x'_1 = J_v$$

Combining (7.18) and (7.19) shows (7.16).

### Completion of the proof of Proposition 6.1.

In hypothesis (7.1) we supposed  $r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1)$ . This forces  $J$  and  $J' = \mathcal{B} \cdot J$  to have the same ideal class as fractional  $k$ -ideals. Hence  $\mathcal{B} = O \cdot \lambda$  is a principal ideal for some  $\lambda \in k^*$ . With the notation of Corollaries 7.4 and 7.5, we now see that if we choose  $x_v = \lambda$  for all places  $v$ , then the matrix  $M' = \lambda^{-1} \cdot M \in \text{GL}_2(k)$  has image  $M_v$  in  $\text{GL}_2(k_v)$  for all  $v$ . This implies  $M' \in \Gamma_0(S, S')$ . Since  $M$  and  $M'$  have the same image  $\sigma$  in  $\text{PGL}_2(k)$ , we have  $\sigma \in \Gamma_0(S, S')$ , which completes the proof of Proposition 6.1 by Lemma 7.1.

## 8. PROOF OF PROPOSITION 6.2.

Let  $\mathcal{D}'$  be the maximal  $O$ -order  $\cap_{v \notin S'} \mathcal{D}_v \cap_{v \in S'} \mathcal{D}'_v$  in  $\text{Mat}_2(k)$ , where  $\mathcal{D}_v$  and  $\mathcal{D}'_v$  are defined in §2. The set  $R_f$  of finite places of  $k$  which ramify in  $\text{Mat}_2(k)$  is empty. Therefore the group  $\Gamma_{R_f}$  which Borel defines in [3, §8.4] is the image in  $\text{PGL}_2(k)$  of the group  $B_{R_f}^*$  of elements  $\tau \in \text{GL}_2(k)$  such that  $\det(\tau) \in O^*$ . Define  $\Gamma_{\mathcal{D}'^*}$  (resp. to  $\Gamma_{\mathcal{D}'_1}$ ) to be the image in  $\text{PGL}_2(k)$  of  $\mathcal{D}'^*$  (resp. the image of the group of  $\tau \in \mathcal{D}'^*$  such that  $\det(\tau) = 1$ .) Borel shows in [3, Lemma 8.5] that  $[\Gamma_{R_f} : \Gamma_{\mathcal{D}'_1}] = 2$ , since in our case the unit group  $O^*$  is finite, cyclic and of even order and  $k$  has no real places. However, we also have  $[\Gamma_{\mathcal{D}'^*} : \Gamma_{\mathcal{D}'_1}] = 2$ , since  $\mathcal{D}'^*$  contains a diagonal matrix whose diagonal entries are 1 and a generator of  $O^*$ . Since

$$\Gamma_{\mathcal{D}'_1} \subset \Gamma_{\mathcal{D}'^*} \subset \Gamma_{R_f}$$

we conclude that  $\Gamma_{\mathcal{D}'^*} = \Gamma_{R_f}$ . Hence Borel's result in the Lemma of [3, §8.6] shows

$$(8.1) \quad [\Gamma_{\mathcal{D}'} : \Gamma_{\mathcal{D}'^*}] = h_{2,k}$$

where  $\Gamma_{\mathcal{D}'}$  is the image in  $\mathrm{PGL}_2(k)$  of the normalizer  $\mathrm{Norm}(\mathcal{D}')$  of  $\mathcal{D}'$  in  $\mathrm{GL}_2(k)$ . For  $v \in S$ , let  $k(v) = O_v/\pi_v O_v$ , and let  $b(v)$  be the subgroup of lower triangular matrices in  $\mathrm{GL}_2(k(v))$ . Definition 5.1 implies that  $\Gamma_0(S, S')$  is the image in  $\mathrm{PGL}_2(k)$  of the subgroup  $\mathcal{D}'(S)^*$  of elements  $M \in \mathcal{D}'^*$  such that the image of  $M$  in

$$\mathcal{D}'_v/\pi_v \mathcal{D}'_v = \mathcal{D}_v/\pi_v \mathcal{D}_v = \mathrm{Mat}_2(O_v/\pi_v O_v)$$

lies in  $b(v)$  for each  $v \in S$ . Since each of the  $1 + \#k(v)$  cosets of  $b(v)$  in  $\mathrm{GL}_2(k(v))$  is represented by an element of  $\mathrm{SL}_2(k(v))$ , the Strong Approximation Theorem for  $\mathrm{SL}_2$  implies

$$(8.2) \quad [\mathcal{D}'^* : \mathcal{D}'(S)^*] = \prod_{v \in S} (1 + \#k(v)).$$

Clearly  $\mathcal{D}'^* \cap k^* = \mathcal{D}'(S)^* \cap k^*$  when we identify these groups with the diagonal matrices inside  $\mathcal{D}'^*$  and  $\mathcal{D}'(S)^*$ . Thus (8.2) gives

$$(8.3) \quad [\Gamma_{\mathcal{D}'^*} : \Gamma_0(S, S')] = \prod_{v \in S} (1 + \#k(v)).$$

The group  $\Gamma_{\mathcal{D}'}$  is equal to  $\Gamma_{\emptyset, S'}$  by [3, §4.9, eq. (4)]. Hence on letting

$$\Gamma_2 = \Gamma_{\mathcal{D}'} \cap \Gamma_{S, S'} = \Gamma_{\emptyset, S'} \cap \Gamma_{S, S'}$$

we have from [3, §5.3, eq. (7) and (8)] that

$$(8.4) \quad [\Gamma_{\mathcal{D}'} : \Gamma_2] = \prod_{v \in S} (1 + \#k(v))$$

(Note that there is a misprint in [3, §5.3, eq. (4)], since the product in that equation should be over places in  $S$ .) Putting together (8.1), (8.3) and (8.4) gives the generalized index relation

$$(8.5) \quad [\Gamma_2 : \Gamma_0(S, S')] = [\Gamma_{\mathcal{D}'} : \Gamma_{\mathcal{D}'^*}] \cdot [\Gamma_{\mathcal{D}'^*} : \Gamma_0(S, S')]/[\Gamma_{\mathcal{D}'} : \Gamma_2] = h_{2, k}.$$

We now define a homomorphism

$$(8.6) \quad F : \Gamma_{S, S'} \rightarrow \prod_{v \in S} (\mathbf{Z}/2)$$

by sending  $\sigma \in \Gamma_{S, S'}$  to the vector having component 0 at  $v \in S$  if  $\sigma$  is even at  $v$  and component 1 if  $v$  is odd at  $v$ . The kernel of  $F$  is

$$\Gamma_2 = \Gamma_{\emptyset, S'} \cap \Gamma_{S, S'}$$

so

$$(8.7) \quad [\Gamma_{S, S'} : \Gamma_2] = \#\mathrm{Image}(F)$$

Consider the homomorphism

$$(8.8) \quad T : \prod_{v \in S} (\mathbf{Z}/2) \rightarrow \mathrm{Cl}(k)/(2\mathrm{Cl}(k))$$

which sends the vector having component 1 at  $v$  and component 0 at the other places in  $S$  to the class of the prime ideal  $\mathcal{P}(v)$ . We will show that

$$(8.9) \quad \mathrm{Image}(F) = \mathrm{Kernel}(T).$$

Before proving (8.9) note that in the statement of Proposition 6.2,  $\mathrm{Image}(T)$  has order  $2^n$ . Thus (8.7) and (8.9) will show

$$(8.10) \quad [\Gamma_{S, S'} : \Gamma_2] = \#\mathrm{Image}(F) = \#\mathrm{Kernel}(T) = 2^{\#S}/\#\mathrm{Image}(T) = 2^{\#S-n}$$

Hence (8.5) and (8.10) prove (6.1), which will prove Proposition 6.2.

It remains to show (8.9). If  $M \in \mathrm{GL}_2(k)$  represents  $\sigma \in \Gamma_{S, S'}$ , then  $\det(M) \in k^*$  is even at all  $v \notin S$ , and  $\mathrm{ord}_v(\det(M))$  is even (resp. odd) exactly if the component of  $F(\sigma)$  at  $v$  is 0 (resp. 1). Since  $\det(M)$  generates a principal ideal, it follows that the composition  $T \circ F$  is trivial, so  $\mathrm{Image}(F) \subset \mathrm{Kernel}(T)$ .

To show equality in (8.9), it will now suffice to show the following. Suppose  $\lambda \in k^*$  has  $\text{ord}_v(\lambda) \equiv 0 \pmod{2\mathbf{Z}}$  for  $v \notin S$ . Then we need to show there is an element  $\sigma \in \Gamma_{S,S'}$  which for  $v \in S$  is odd at  $v$  if and only if  $\text{ord}_v(\lambda)$  is odd. Without loss of generality, we can assume  $\lambda \in O$ . Fix a uniformizing element  $\pi_v \in O_v$  for each place  $v$ . We can choose an element  $c \in O$  satisfying the following finite system of congruences:

$$(8.11) \quad \text{If } \text{ord}_v(\lambda) = 2a_v + 1 \text{ is odd, then } c \equiv 0 \pmod{\pi_v^{a_v+1}O_v};$$

$$(8.12) \quad \text{If } \text{ord}_v(\lambda) = 2a_v \text{ is even, and } a_v > 0 \text{ or } v \in S \cup S', \text{ then } c \equiv \lambda \cdot \pi_v^{-a_v} - \pi_v^{a_v} \pmod{\pi_v^{a_v+1}O_v}.$$

We let  $\sigma'$  be the matrix

$$(8.13) \quad \sigma' = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix}.$$

Clearly  $\det(\sigma') = -\lambda$  is odd at exactly the places  $v$  of  $k$  where  $\text{ord}_v(\lambda)$  is odd. We will show below that

$$(8.14) \quad \sigma' \text{ fixes an edge of } T_v \text{ if } v \in S \cup S' \text{ or } \text{ord}_v(\lambda) \neq 0.$$

Let us first prove that (8.14) implies equality in (8.9), which we have already proved will complete the proof of Proposition 6.2. If  $v \in S$ , then (8.14) states that  $\sigma'$  fixes an edge of  $T_v$ . If  $v \in S'$  or  $\text{ord}_v(\lambda) \neq 0$ , then  $\sigma'$  must fix an edge of  $T_v$  pointwise, since (8.14) says  $\sigma'$  fixes an edge, and  $\sigma'$  is even at  $v$ . Finally, if  $v \notin S \cup S'$  and  $\text{ord}_v(\lambda) \neq 0$ , then  $\sigma'$  lies in  $\text{GL}_2(O_v)$ , so  $\sigma'$  fixes the vertex of  $T_v$  that is fixed by every element of  $\Gamma_{S,S'}$ . The group  $\text{SL}_2(k_v)$  acts transitively on the edges of  $T_v$ . Hence we can conclude from the Strong Approximation Theorem that a conjugate  $\sigma$  of  $\sigma'$  by an element of  $\text{SL}_2(k)$  defines an element of  $\Gamma_{S,S'}$ . Since  $\det(\sigma) = \det(\sigma')$  has odd valuation at exactly those  $v$  where  $\text{ord}_v(\lambda)$  is odd, this implies equality holds in (8.9).

We now prove (8.14). Suppose first that  $v$  is a place for which  $\text{ord}_v(\lambda) = 2a_v + 1$  is odd, so that  $v$  lies in  $S$ . Condition (8.11) implies that  $\sigma'$  acts in the following way on the lattices  $\pi_v^{a_v}O_v \oplus O_v$  and  $\pi_v^{a_v+1}O_v \oplus O_v$  in  $k_v \oplus k_v$ .

$$(8.15) \quad \sigma' \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} \lambda O_v \\ \pi_v^{a_v}O_v \end{pmatrix} = \pi_v^{a_v} \begin{pmatrix} \pi_v^{a_v+1}O_v \\ O_v \end{pmatrix}.$$

$$(8.16) \quad \sigma' \begin{pmatrix} \pi_v^{a_v+1}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v+1}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} \lambda O_v \\ \pi_v^{a_v+1}O_v \end{pmatrix} = \pi_v^{a_v+1} \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix}.$$

These equalities show that  $\sigma'$  interchanges the homothety classes of  $\pi_v^{a_v}O_v \oplus O_v$  and  $\pi_v^{a_v+1}O_v \oplus O_v$ . Hence  $\sigma'$  fixes the edge of  $T_v$  between these homothety classes (though it clearly does not fix this edge pointwise).

Now suppose that  $\text{ord}_v(\lambda) = 2a_v$  is even and  $a_v > 0$  or  $v \in S \cup S'$ . In all cases we have  $a_v \geq 0$ , since  $\lambda \in O$ . Condition 8.12 implies  $c \equiv 0 \pmod{\pi_v^{a_v}O_v}$ . Hence

$$(8.17) \quad \sigma' \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} \lambda O_v \\ \pi_v^{a_v}O_v \end{pmatrix} = \pi_v^{a_v} \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix}.$$

Thus  $\sigma'$  fixes the homothety class of  $\pi_v^{a_v}O_v \oplus O_v$ , so it will now suffice to show  $\sigma'$  fixes the homothety class of an  $O_v$  lattice  $L$  containing  $\pi_v^{a_v}O_v \oplus O_v$  for which  $L/(\pi_v^{a_v}O_v \oplus O_v)$  is  $O_v$ -isomorphic to  $k(v) = O_v/\pi_v O_v$ . We now compute

$$(8.18) \quad \begin{aligned} \sigma' \begin{pmatrix} \pi_v^{a_v-1} \\ \pi_v^{-1} \end{pmatrix} &= \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v-1} \\ \pi_v^{-1} \end{pmatrix} = \begin{pmatrix} \lambda \pi_v^{-1} \\ \pi_v^{a_v-1} + c \pi_v^{-1} \end{pmatrix} \\ &= \pi_v^{a_v} \cdot \begin{pmatrix} \lambda \pi_v^{-2a_v} \pi_v^{a_v-1} \\ (1 + c \pi_v^{-a_v}) \pi_v^{-1} \end{pmatrix} \\ &\equiv \pi_v^{a_v} \cdot \lambda \pi_v^{-2a_v} \cdot \begin{pmatrix} \pi_v^{a_v-1} \\ \pi_v^{-1} \end{pmatrix} \pmod{\pi_v^{a_v} \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix}}. \end{aligned}$$

where the last congruence results from the condition on  $c$  in (8.12). Here  $\lambda \cdot \pi_v^{-2a_v}$  is a unit of  $O_v$ , so we can take the lattice  $L$  to be the one generated as an  $O_v$ -module by  $\pi_v^{a_v} O_v \oplus O_v$  and the vector  $(\pi_v^{a_v-1}, \pi_v^{-1})$ . This completes the proof of Proposition 6.2.

### 9. CUSPS AND CLASS NUMBERS.

We begin by giving an ineffective proof of part (1) of Corollary 1.2. Recall that  $C(N)$  is the set of isometry classes of minimal finite covolume discrete arithmetic hyperbolic 3-orbifolds having exactly  $N$  cusps. The finite covolume discrete arithmetic subgroups of  $\mathrm{PGL}_2(k)$  are commensurable. Hence to show the elements of  $C(N)$  represent only finitely many distinct commensurability classes, it will suffice by Theorem 1.1 to show that there are only finitely many imaginary quadratic fields  $k$  such that  $h_k/h_{k,2} \leq N$ . Siegel proved in [8] that for each  $\epsilon > 0$ , there is an ineffective constant  $c(\epsilon) > 0$  such that

$$(9.1) \quad h_k > c(\epsilon) |d_k|^{\frac{1}{2}-\epsilon}$$

where  $d_k$  is the discriminant of  $k$ . By a result of Tatzawa, the constant  $c(\epsilon)$  can be made effective except for at most one exceptional field  $k$ ; see [9] and [5]. Let  $n_k$  be the number of distinct prime factors of  $d_k$ . By genus theory, the two-rank of the ideal class group of  $k$  is equal to  $2^{n_k-1}$ . Thus  $h_{2,k} = 2^{n_k-1}$  and we get

$$(9.2) \quad \frac{h_k}{h_{k,2}} > c(\epsilon) \frac{|d_k|^{\frac{1}{2}-\epsilon}}{2^{n_k-1}} > c(\epsilon) \prod_{p|d_k} \frac{p^{\frac{1}{2}-\epsilon}}{2}.$$

The fact that there are only finitely many  $k$  for which  $h_k/h_{k,2} < N$  is clear from (9.2), since  $-d_k$  is either a square-free positive integer or 4 times such an integer, and if  $\epsilon < 1/2$  then there are only finitely many primes  $p$  for which  $\frac{p^{\frac{1}{2}-\epsilon}}{2} < 2$ .

Suppose now that  $X$  is an element of  $C(N)$ . To show part (2) of Corollary 1.2, we must show that there are infinitely many elements of the commensurability class of  $X$  which also lie in  $C(N)$ . By Borel's work,  $X = \mathbf{H}^3/\Gamma_{S,S'}$  for some imaginary quadratic field  $k$  and some maximal discrete subgroup  $\Gamma_{S,S'} \subset \mathrm{PGL}_2(k)$ . Theorem 1.1 shows

$$(9.3) \quad 2^n \frac{h_k}{h_{k,2}} = N$$

where  $2^n$  is the order of the subgroup of  $Cl(k)/2Cl(k)$  generated by the places in  $S$ . We now let  $S_0$  be a set of  $n$  places whose images in  $Cl(k)/2Cl(k)$  generate a subgroup of order  $2^n$ ; such an  $S_0$  exists by the Chebotarev density theorem. Let  $W$  be the set of finite places  $v$  of  $k$  such that

$$(9.4) \quad \mathcal{P}(v) \cdot \mathcal{P}(S_0) = \mathcal{P}(S_0 \cup \{v\})$$

is principal. The Chebotarev density theorem also implies  $W$  is infinite. We claim that for  $v \in W$ , the group  $\Gamma_{S_0 \cup \{v\}, \emptyset}$  contains an element  $\sigma_v$  which is odd at  $S_0 \cup \{v\}$ . To construct  $\sigma_v$ , let  $\lambda_v$  be a generator for the ideal in (9.4). We can then take  $\sigma_v$  to be

$$(9.5) \quad \sigma_v = \begin{pmatrix} 0 & \lambda_v \\ 1 & 0 \end{pmatrix}.$$

We now see from [3, Prop. 4.4(iii)] that if  $\Gamma_{S_0 \cup \{v\}, \emptyset}$  is not maximal, then it is conjugate to subgroup of a maximal discrete subgroup of the form  $\Gamma_{S_0 \cup \{v\}, S'(v)}$  for some finite set of places  $S'(v)$  which is disjoint from  $S_0 \cup \{v\}$ . We conclude that for each  $v \in W$ , there is a maximal discrete group  $\Gamma_{S_0 \cup \{v\}, S'(v)}$  which contains an element which is odd at  $v$ . Furthermore, the fact that (9.4) is principal implies that  $\{\mathcal{P}(v') : v' \in S_0\}$  and  $\{\mathcal{P}(v') : v' \in S_0\} \cup \{\mathcal{P}(v)\}$  generate the same subgroup of  $Cl(k)/2Cl(k)$ , which by hypothesis has order  $2^n$ . Thus Theorem 1.1 shows  $\mathbf{H}^3/\Gamma_{S_0 \cup \{v\}, S'(v)}$  has exactly  $N$  cusps, were  $N$  is as in (9.3). The orbifolds  $\mathbf{H}^3/\Gamma_{S_0 \cup \{v\}, S'(v)}$  and  $\mathbf{H}^3/\Gamma_{S_0 \cup \{v'\}, S'(v')}$  are not isometric for distinct  $v$  and  $v'$  in  $W - S_0$ , since the group  $\Gamma_{S_0 \cup \{v\}, S'(v)}$  contains no elements

which are odd at  $v'$  and similarly with the roles of  $v$  and  $v'$  reversed (cf. [3, Prop. 4.4(ii)]). This completes the proof that  $C(N)$  contains infinitely many elements which are commensurable to  $X$ .

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