PULLBACK MODULI SPACES

FRAUKE M. BLEHER AND TED CHINBURG

Abstract. Geometric invariant theory can be used to construct moduli spaces associated to representations of finite dimensional algebras. One difficulty which occurs in various natural cases is that non-isomorphic modules are sent to the same point in the moduli spaces which arise. In this paper we study how this collapsing phenomenon can sometimes be reduced by considering pullbacks of modules for an auxiliary algebra. One application is a geometric proof that the twisting action of an algebra automorphism induces an algebraic isomorphism between moduli spaces.

1. Introduction

To construct a well-defined moduli space associated to algebraic objects of various kinds, one must introduce an equivalence relation on these objects and classify the resulting equivalence classes. These equivalence classes may be much larger than one would like, so that many isomorphism classes of objects are collapsed to a point in the moduli space. A standard approach to dealing with this problem is to incorporate more structure into the moduli problem. In this paper we consider moduli spaces constructed by King in a now classical paper [10] for representations of finite dimensional algebras Λ over an algebraically closed field. Our approach to adding more structure is to introduce an algebra homomorphism $\Lambda \to \Lambda'$ and to consider families of Λ-modules which are pullbacks of families of Λ'-modules. The main problem then is to control the loss of information resulting from restricting families of Λ'-modules to families of Λ-modules. For certain Λ', we will produce sufficient conditions on $\Lambda \to \Lambda'$ for no information to be lost in the pullback process. This leads in various cases to non-trivial moduli spaces when the moduli spaces which result from applying King's methods directly to Λ-modules alone collapse to points (see Example 2.4 and §3). A different approach to these moduli problems has recently been developed by Huisgen-Zimmermann in [8, 9]. She uses closed subvarieties of Grassmannians to describe the degenerations of a particular Λ-module. Under certain conditions she obtains moduli spaces for all isomorphism classes of Λ-modules with a given top and dimension.

We now introduce the notation needed to state our results. Let K be an algebraically closed field, and suppose $\Lambda = KQ/J$ is a finite dimensional basic algebra

1991 *Mathematics Subject Classification.* Primary 16G10; Secondary 14D22, 16W20.

Key words and phrases. Moduli spaces, pullbacks, representations of finite dimensional algebras, algebra automorphisms.

The first author was supported in part by NSF Grant DMS01-39737 and by NSA Grant H98230-06-1-0021. The second author was supported in part by NSF Grants DMS00-70433 and DMS05-00106.

over K, i.e. Q is a finite quiver and J is an admissible ideal in the path algebra KQ. For each such Λ and dimension vector d, there is a representation variety $V_\Lambda(d)$, whose points correspond to representations of Λ with this dimension vector (see, for example, [10, 12, 15] and their references). The isomorphism classes of these Λ-modules correspond to orbits of the variety $V_\Lambda(d)$ under the action of a reductive algebraic group, called $\mathrm{GL}(d)$. Let Λ' be another basic finite dimensional K-algebra and let $f : \Lambda \to \Lambda'$ be a K-algebra homomorphism.

King's work in [10] can be applied to construct a fine moduli space $\mathcal{M}^s_{\Lambda'}(d', \theta')$ for families of Λ'-modules of dimension vector d' which are stable with respect to an additive function $\theta' : K_0(\Lambda') \to \mathbb{Z}$. We will consider the functor which associates to a variety X over K the set of equivalence classes of families of Λ-modules which are pullbacks via f of families of θ'-stable Λ'-modules (see Definition 2.5). The notion of equivalence used here is fiberwise isomorphism. Our first main result, Theorem 2.7, is that this functor has a fine moduli space $\mathcal{M}^s_{f,d',\theta'}$, provided the pullbacks via f of non-isomorphic θ'-stable Λ'-modules are non-isomorphic Λ-modules. One application is to show in Theorem 2.10 that under the hypotheses of Theorem 2.7, the twisting action of an algebra automorphism σ of Λ induces an algebraic isomorphism between $\mathcal{M}^s_{f,d',\theta'}$ and $\mathcal{M}^s_{f\circ\sigma,d',\theta'}$.

We will analyze two types of Λ' and then present some examples suggesting further generalizations.

In §3 we specialize to the case in which Λ' is the path algebra KQ' of a circular quiver Q'. We call the Λ-modules which arise from this choice circular modules. Many authors have used circular quivers to construct indecomposable Λ-modules (see, for example, [7, 14, 6, 18, 5]). We show in §3.1 how circular modules generalize some of these results. In Theorem 3.8, we find sufficient conditions on the algebra homomorphism $f : \Lambda \to \Lambda'$ for the fine moduli spaces $\mathcal{M}^s_{f,d',\theta'}$ of Theorem 2.7 to be well-defined. We then specialize further in §3.2 to the case in which Λ is a special biserial algebra, for which the indecomposable Λ-modules are well-known. In this case the circular modules are band modules (see [5] or [11] for the definition of band modules). By specializing Theorem 2.10 to this case we obtain an algebro-geometric proof of a result proved earlier by one of us [3, 4] that twisting band modules by automorphisms of special biserial algebras gives rise to algebraic automorphisms of the parameter spaces of these modules. In §3.3, we discuss some other algebras for which not all circular modules are band modules.

In §4 we define multi-strand modules, which arise from quivers associated to canonical algebras. When Λ' is an algebra giving rise to such modules, we analyze in Theorem 4.7 some sufficient conditions on $f : \Lambda \to \Lambda'$ for the fine moduli spaces $\mathcal{M}^s_{f,d',\theta'}$ of Theorem 2.7 to be well-defined. When $\mathcal{M}^s_{f,d',\theta'}$ is well-defined we show that it is a projective space whose dimension may be larger than 1.

In a final section we will give an example in which the moduli space $\mathcal{M}^s_{f,d',\theta'}$ is $\mathbb{P}^r \times \mathbb{P}^s$, King's moduli space for the associated Λ-modules is $\mathbb{P}^s$, and the natural map from $\mathcal{M}^s_{f,d',\theta'}$ to King's moduli space $\mathbb{P}^s$ collapses the first factor $\mathbb{P}^r$ of $\mathcal{M}^s_{f,d',\theta'}$. There are many other examples of algebras Λ for which there are known families of indecomposable Λ-modules parameterized by particular algebraic varieties, e.g. by projective spaces. One natural question at this point is to consider when such families arise from the moduli spaces constructed in this paper for well chosen algebras Λ' and algebra homomorphisms $\Lambda \to \Lambda'$.

Throughout this article, K denotes an algebraically closed field and K^* denotes the set of non-zero elements of K. All algebras are finite dimensional K-algebras, all modules are finitely generated left modules. Λ-mod denotes the category of finitely generated Λ-modules. For precise definitions of quivers, relations and path algebras see e.g. [1].

2. Moduli spaces

In this section we describe a variation of the moduli spaces defined by King in [10]. We first recall a few of the definitions from [10].

Suppose $\Lambda = KQ/J$ is an arbitrary basic algebra, and $\mathcal{C}$ is the category of connected varieties (over K) in the sense of King [10], Newstead [13] and Serre [17].

Definition 2.1. Let $\theta : K_0(\Lambda) \to \mathbb{Z}$ be an additive function, where $K_0(\Lambda)$ denotes the Grothendieck group which has as $\mathbb{Z}$-basis the isomorphism classes of simple Λ-modules.

(i) A Λ-module M is called θ-semistable (resp. θ-stable), if $\theta([M]) = 0$ and every proper non-zero submodule M' of M satisfies $\theta([M']) \geq 0$ (resp. $\theta([M']) > 0$).

(ii) Let Λ-mod$^{\theta,ss}$ be the full subcategory of θ-semistable modules in Λ-mod. The simple objects in this subcategory are the θ-stable modules. Since Λ-mod$^{\theta,ss}$ is both artinian and noetherian, every θ-semistable module M has a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

such that M_i/M_{i-1} is a θ-stable module for all i. Define

$$\mathrm{gr}(M) = \bigoplus_{i=1}^{n} (M_i/M_{i-1}).$$

By [16, Theorem 2.1], $\mathrm{gr}(M)$ does not depend on the filtration of M.

(iii) Two θ-semistable Λ-modules M and M' are called S-equivalent, if $\mathrm{gr}(M)$ is isomorphic to $\mathrm{gr}(M')$. In particular, two θ-stable modules are S-equivalent if and only if they are isomorphic.

For the moduli spaces we also need the definition of families of Λ-modules.

Definition 2.2. A family of Λ-modules over a variety X in $\mathcal{C}$ is a locally free sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules together with a K-algebra homomorphism $\alpha_{\mathcal{F}} : \Lambda \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$. A family $\mathcal{F}$ over X is called a family of θ-semistable (resp. θ-stable) Λ-modules, if for every point $x \in X$ the fiber $\mathcal{F}_x$ is a θ-semistable (resp. θ-stable) Λ-module. The notion of S-equivalence can be extended to families. We say that two families $\mathcal{F}$ and $\mathcal{F}'$ of θ-semistable Λ-modules over X are S-equivalent if for all $x \in X$ the fibers $\mathcal{F}_x$ and $\mathcal{F}'_x$ are S-equivalent. This extension of equivalence to families arises in [13, Prop. 1.8].

Let now d be a dimension vector, i.e. $d \in K_0(\Lambda)$ with non-negative coefficients. Then we have the following result due to King [10].

Theorem 2.3. (King)

(i) *There exists a coarse moduli space $\mathcal{M}_\Lambda(d,\theta)$ for families of θ-semistable Λ-modules of dimension vector d, up to S-equivalence. Moreover, $\mathcal{M}_\Lambda(d,\theta)$ is a projective variety.*

(ii) *The moduli space $\mathcal{M}_\Lambda(d,\theta)$ contains an open set $\mathcal{M}^s_\Lambda(d,\theta)$ whose points correspond to isomorphism classes of θ-stable Λ-modules of dimension vector d. In particular, $\mathcal{M}^s_\Lambda(d,\theta)$ is a quasi-projective variety. If d is an indivisible vector, then $\mathcal{M}^s_\Lambda(d,\theta)$ is a fine moduli space for families of θ-stable Λ-modules of dimension vector d, up to isomorphism.*

Theorem 2.3 means the following. For part (i) (resp. part (ii)) consider the contravariant functor

$$F : \mathcal{C} \to \mathrm{Sets}$$

which sends each object X in $\mathcal{C}$ to the set $F(X)$ of all S-equivalence classes of families of θ-semistable (resp. θ-stable) Λ-modules over X of dimension vector d. A morphism $\phi : X' \to X$ in $\mathcal{C}$ is sent to $F(\phi) : F(X) \to F(X')$ with $F(\phi)([\mathcal{F}]) = [\phi^*(\mathcal{F})]$.

In case $\mathcal{M} = \mathcal{M}^s_\Lambda(d,\theta)$ is a fine moduli space, F is represented by $\mathcal{M}$; more precisely, F is isomorphic to the functor $\mathrm{Hom}_\mathcal{C}(-,\mathcal{M})$. In particular, there exists a universal family $\mathcal{U}$ over $\mathcal{M}$ which corresponds to the identity morphism $\mathrm{id}_\mathcal{M}$ such that for every family $\mathcal{F}$ over X, there exists a unique morphism $\mu : X \to \mathcal{M}$ with $[\mathcal{F}] = [\mu^*(\mathcal{U})]$.

In case $\mathcal{M} = \mathcal{M}_\Lambda(d,\theta)$ is a coarse moduli space, there is a natural transformation $\Phi : F \to \mathrm{Hom}_\mathcal{C}(-,\mathcal{M})$ such that $\Phi(\mathrm{pt})$ is bijective and Φ satisfies the natural universal property with respect to natural transformations $\Psi : F \to \mathrm{Hom}_\mathcal{C}(-,N)$ for arbitrary varieties N in $\mathcal{C}$.

The points of $\mathcal{M}_\Lambda(d,\theta)$ (resp. $\mathcal{M}^s_\Lambda(d,\theta)$) are in bijection with the S-equivalence classes of θ-semistable (resp. θ-stable) Λ-modules of dimension vector d.

The following example illustrates how non-isomorphic indecomposable modules can correspond to the same point of the moduli spaces of Theorem 2.3, even though these modules correspond bijectively to the points of an algebraic variety over K.

Example 2.4. Let Λ be the algebra $\Lambda = K\langle\alpha,\beta\rangle/(\alpha^2,\beta^2,\alpha\beta-\beta\alpha)$, which is special biserial. Since Λ has a unique simple module T up to isomorphism, the only possible additive functions $\theta : K_0(\Lambda) \to \mathbb{Z}$ are given by $\theta([T]) = a$, $a \in \mathbb{Z}$. If $a \neq 0$, the zero module is the only θ-semistable Λ-module. Suppose now $a = 0$. Then all Λ-modules are θ-semistable and T is the only θ-stable module, up to isomorphism. Consider the band modules $M(\beta\alpha^{-1},\lambda,1)$ defined for $\lambda \in K \cup \{\infty\}$ in the following way (see [5, §3]). Define $M(\beta\alpha^{-1},\lambda,1)$ to be the two dimensional vector space on which α acts as the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and β acts as $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ if $\lambda \neq \infty$. In case $\lambda = \infty$, α acts as the zero matrix and β acts as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. These band modules are not isomorphic for any two values of λ that define different points on the projective line $\mathbb{P}^1$ over K. Nonetheless, they are all S-equivalent. Hence they are not distinguished by King's moduli space $\mathcal{M}_\Lambda(d,\theta)$ for $d = 2$. However, the methods we will now describe together with the results in §3 will enable us to define a fine moduli space for these band modules, up to isomorphism (see Theorem 3.8 and Proposition 3.11).

We now describe a variation of King's construction. We need the following definition.

Definition 2.5. Suppose Λ' is a basic finite dimensional K-algebra and that there is an algebra homomorphism $f : \Lambda \to \Lambda'$. Let $\theta' : K_0(\Lambda') \to \mathbb{Z}$ be an additive function, and let d' be a dimension vector in $K_0(\Lambda')$. Suppose X is a variety in $\mathcal{C}$, and $\mathcal{G}$ (resp. $\mathcal{F}$) is a family of Λ-modules (resp. Λ'-modules) over X.

(i) We call $\mathcal{G}$ the pullback via f of $\mathcal{F}$, written $\mathcal{G} = f^*\mathcal{F}$, if $\mathcal{G} = \mathcal{F}$ as locally free sheaves of $\mathcal{O}_X$-modules and $\alpha_{\mathcal{G}} = \alpha_{\mathcal{F}} \circ f$, where $\alpha_{\mathcal{F}} : \Lambda' \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$ (resp. $\alpha_{\mathcal{G}} : \Lambda \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{G})$) is the K-algebra homomorphism which defines the Λ'-action on $\mathcal{F}$ (resp. the Λ-action on $\mathcal{G}$), as described in Definition 2.2.

If all the Λ'-modules in the family $\mathcal{F}$ have the same dimension vector d', then all the Λ-modules in the pullback family $\mathcal{G} = f^*\mathcal{F}$ have the same dimension vector in $K_0(\Lambda)$ which we will denote by d'_f.

(ii) We call $\mathcal{G}$ a family of θ'_f-semistable (resp. θ'_f-stable) Λ-modules, if there exists a family $\mathcal{F}$ of θ'-semistable (resp. θ'-stable) Λ'-modules such that $\mathcal{G}$ is fiberwise isomorphic to $f^*\mathcal{F}$, where $f^*\mathcal{F}$ is the pullback of $\mathcal{F}$ as defined in part (i).

(iii) Suppose $\mathcal{G}$ and $\mathcal{G}'$ are two families of θ'_f-semistable (resp. θ'_f-stable) Λ-modules as defined in part (ii). We call $\mathcal{G}$ and $\mathcal{G}'$ S_f-equivalent if there are two S-equivalent families $\mathcal{F}$ and $\mathcal{F}'$ over X of θ'-semistable (resp. θ'-stable) Λ'-modules such that $\mathcal{G}$ is fiberwise isomorphic to $f^*\mathcal{F}$ and $\mathcal{G}'$ is fiberwise isomorphic to $f^*\mathcal{F}'$.

(iv) We say f is θ'-semistably separated (resp. θ'-stably separated) with respect to d', if whenever N and N_0 are θ'-semistable (resp. θ'-stable) Λ'-modules of dimension vector d' with $f^*N \cong f^*N_0$ as Λ-modules then N and N_0 are S-equivalent.

(v) Let $\mathcal{M}_{\Lambda'}(d', \theta')$ (resp. $\mathcal{M}^s_{\Lambda'}(d', \theta')$) be the moduli space from Theorem 2.3 for families of θ'-semistable (resp. θ'-stable) Λ'-modules of dimension vector d', up to S-equivalence. In case f is θ'-semistably separated (resp. θ'-stably separated) with respect to d', we define $\mathcal{M}_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f,d',\theta'}$) to be the projective variety $\mathcal{M}_{\Lambda'}(d', \theta')$ (resp. the quasi-projective variety $\mathcal{M}^s_{\Lambda'}(d', \theta')$).

Remark 2.6. (i) We will regard single Λ-modules (resp. Λ'-modules) as families of modules over a point. In particular, parts (i), (ii) and (iii) of Definition 2.5 pertain to a single Λ-module $\mathcal{G}$ (resp. a single Λ'-module $\mathcal{F}$).

(ii) In Definition 2.5 we have not chosen an additive function $\theta : K_0(\Lambda) \to \mathbb{Z}$. One would first have to choose such a θ to be able to consider the θ-stability, θ-semistability and S-equivalence classes of Λ-modules. It is a natural problem to find when one can choose a θ so that if $\mathcal{G}$ and $\mathcal{G}'$ as in part (iii) of Definition 2.5 are families of θ'_f-semistable (resp. θ'_f-stable) Λ-modules and S_f-equivalent, then they are families of θ-semistable (resp. θ-stable) Λ-modules and S-equivalent. We address in Remark 2.8 the question of when there is a morphism from the varieties in part (v) of Definition 2.5 to King's moduli spaces associated with a choice of θ.

(iii) As in Definition 2.1(iii), two θ'-stable Λ'-modules are S-equivalent if and only if they are isomorphic. This is not necessarily true for θ'-semistable Λ'-modules. Thus in Definition 2.5(iv), even if f is θ'-semistably separated, S_f-equivalent families $\mathcal{G}$ and $\mathcal{G}'$ as in Definition 2.5(iii) need not be fiberwise isomorphic. However, if f is θ'-semistably separated and $\mathcal{F}$ and $\mathcal{F}'$ are families of θ'-semistable Λ'-modules which are not S-equivalent, then their pullbacks $f^*\mathcal{F}$ and $f^*\mathcal{F}'$ are families of θ'_f-semistable Λ-modules which are not S_f-equivalent (see the proof of Theorem 2.7 below). In particular, if f is θ'-stably separated and $\mathcal{F}$ and $\mathcal{F}'$ are families of θ'-stable Λ'-modules which are not fiberwise isomorphic, then their pullbacks $f^*\mathcal{F}$ and $f^*\mathcal{F}'$ are families of θ'_f-stable Λ-modules which are not fiberwise isomorphic. Thus to reduce collapsing different isomorphism classes of Λ-modules to a point, one should work with θ'-stably separated f and families of θ'_f-stable Λ-modules.

The following result is our main tool for reducing the collapsing phenomenon described in Example 2.4.

Theorem 2.7. *Let Λ', θ', d', f, θ'_f, d'_f be as in Definition 2.5. Suppose that f is θ'-semistably separated (resp. θ'-stably separated) with respect to d', so that $\mathcal{M}_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f,d',\theta'}$) is defined. Then $\mathcal{M}_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f,d',\theta'}$ if d' is indivisible) is a coarse (resp. fine) moduli space for families of θ'_f-semistable (resp. θ'_f-stable) Λ-modules of dimension vector d'_f, up to S_f-equivalence.*

Proof. We consider the functors

$$\begin{aligned} F', G &: \mathcal{C} \to \text{Sets} \\ (\text{resp. } F'^s, G^s &: \mathcal{C} \to \text{Sets}) \end{aligned}$$

which are defined as follows. The functor F' (resp. F'^s) sends each variety X in $\mathcal{C}$ to the set $F'(X)$ (resp. $F'^s(X)$) of S-equivalence classes of families over X of θ'-semistable (resp. θ'-stable) Λ'-modules of dimension vector d'. The functor G (resp. G^s) is the functor which sends each variety X in $\mathcal{C}$ to the set $G(X)$ (resp.$G^s(X)$) of S_f-equivalence classes of families over X of θ'_f-semistable (resp. θ'_f-stable) Λ-modules of dimension vector d'_f. We have to show that $\mathcal{M}_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f,d',\theta'}$ if d' is indivisible) satisfies the properties of a coarse (resp. fine) moduli space with respect to the functor G (resp. G^s). There are two natural transformations

$$\begin{aligned} \Xi &: F' \to G\,, \\ \Xi^s &: F'^s \to G^s \end{aligned}$$

defined in the following way. For each X in $\mathcal{C}$ define $\Xi(X) : F'(X) \to G(X)$ by $\Xi(X)([\mathcal{F}]) = [f^*\mathcal{F}]$, where $[\mathcal{F}]$ denotes the S-equivalence class of $\mathcal{F}$ and $[f^*\mathcal{F}]$ denotes the S_f-equivalence class of $f^*\mathcal{F}$. Let Ξ^s be the restriction of Ξ to F'^s. We claim that there is a unique natural transformation

$$\Psi : G \to F'$$

with $\Psi(X)([f^*\mathcal{F}]) = [\mathcal{F}]$ for every family $\mathcal{F}$ over X of θ'-semistable Λ'-modules of dimension vector d'. To show Ψ is well-defined, suppose $\mathcal{F}$ and $\mathcal{F}'$ are two families over X of θ'-semistable Λ'-modules of dimension vector d' such that $\mathcal{G} = f^*\mathcal{F}$ and $\mathcal{G}' = f^*\mathcal{F}'$ are S_f-equivalent. This means that there are two S-equivalent families $\mathcal{F}_0$ and $\mathcal{F}'_0$ over X of θ'-semistable Λ'-modules of dimension vector d' such that $\mathcal{G}$ (resp. $\mathcal{G}'$) is fiberwise isomorphic to $f^*\mathcal{F}_0$ (resp. $f^*\mathcal{F}'_0$). We need to show that $\mathcal{F}$ and $\mathcal{F}'$ are S-equivalent. Considering the fibers, we have $\mathcal{G}_x = (f^*\mathcal{F})_x \cong (f^*\mathcal{F}_0)_x$ (resp. $\mathcal{G}'_x = (f^*\mathcal{F}')_x \cong (f^*\mathcal{F}'_0)_x$) as Λ-modules for all $x \in X$. Since f is θ'-semistably separated with respect to d', it follows that $\mathcal{F}_x$ and $(\mathcal{F}_0)_x$ (resp. $\mathcal{F}'_x$ and $(\mathcal{F}'_0)_x$) are S-equivalent for all $x \in X$. This implies that $\mathcal{F}$ and $\mathcal{F}_0$ (resp. $\mathcal{F}'$ and $\mathcal{F}'_0$) are S-equivalent. Thus $\mathcal{F}$ and $\mathcal{F}'$ are S-equivalent, which implies that Ψ is well-defined. It is clear that Ψ is unique, since every element of $G(X)$ has the form $[f^*\mathcal{F}]$ for some family $\mathcal{F}$ over X of θ'-semistable Λ'-modules of dimension vector d'. In a similar way we can define a natural transformation

$$\Psi^s : G^s \to F'^s.$$

Since $(\Psi \circ \Xi)(X)$ and $(\Xi \circ \Psi)(X)$ are each the identity map, Ξ is an isomorphism of functors. Similarly, it follows that Ξ^s is an isomorphism of functors. Hence Theorem 2.7 follows from Theorem 2.3. □

Remark 2.8. Let Λ', θ', d', f, θ'_f, d'_f be as in Theorem 2.7, and suppose that f is θ'-semistably separated (resp. θ'-stably separated) with respect to d'. Suppose there is an additive function $\theta : K_0(\Lambda) \to \mathbb{Z}$ such that if $\mathcal{G}$ and $\mathcal{G}'$ are families of θ'_f-semistable (resp. θ'_f-stable) Λ-modules of dimension vector d'_f and S_f-equivalent, then they are families of θ-semistable (resp. θ-stable) Λ-modules of dimension vector d'_f and S-equivalent.

We consider the functors

$$\begin{aligned} F, G &: \mathcal{C} \to \text{Sets} \\ (\text{resp. } F^s, G^s &: \mathcal{C} \to \text{Sets}) \end{aligned}$$

which are defined as follows. The functor F (resp. F^s) sends each variety X in $\mathcal{C}$ to the set $F(X)$ (resp. $F^s(X)$) of S-equivalence classes of families over X of θ-semistable (resp. θ-stable) Λ-modules of dimension vector d'_f. The functor G (resp. G^s) is the functor which sends each variety X in $\mathcal{C}$ to the set $G(X)$ (resp.$G^s(X)$) of S_f-equivalence classes of families over X of θ'_f-semistable (resp. θ'_f-stable) Λ-modules of dimension vector d'_f.

By the above assumptions, there is a natural transformation of functors $G \to F$ (resp. $G^s \to F^s$). Consider King's moduli space $\mathcal{M}_\Lambda(d'_f, \theta)$ (resp. $\mathcal{M}^s_\Lambda(d'_f, \theta)$) associated to F (resp. F^s) from Theorem 2.3, and the moduli space $\mathcal{M}_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f,d',\theta'}$) associated to G (resp. G^s) from Theorem 2.7. Then the definition of coarse moduli space implies that there is a homomorphism $\mathcal{M}_{f,d',\theta'} \to \mathcal{M}_\Lambda(d'_f, \theta)$ (resp. $\mathcal{M}^s_{f,d',\theta'} \to \mathcal{M}^s_\Lambda(d'_f, \theta)$) in $\mathcal{C}$.

In the next theorem we discuss the action on the fine moduli spaces $\mathcal{M}^s_{f,d',\theta'}$ which are induced by algebra automorphisms of Λ. We first need the following definition.

Definition 2.9. Let σ be a K-algebra automorphism of Λ, and let X be a variety in $\mathcal{C}$. Then σ acts on families of Λ-modules over X by "twisting" as follows. Let $\mathcal{G}$ be a family of Λ-modules over X. Then $\sigma(\mathcal{G})$ is the family $\mathcal{H}$ over X such that $\mathcal{H} = \mathcal{G}$ as locally free sheaves of $\mathcal{O}_X$-modules and $\alpha_{\mathcal{H}} = \alpha_{\mathcal{G}} \circ \sigma$.

Theorem 2.10. *Let Λ', θ', d', f be as in Theorem 2.7. Suppose that d' is indivisible and that f is θ'-stably separated with respect to d'. Let σ be an algebra automorphism of Λ and define $f' = f \circ \sigma$. Let $\mathcal{U}_f$ (resp. $\mathcal{U}_{f'}$) be the universal family over $\mathcal{M}^s_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f',d',\theta'}$). Then there exists an isomorphism $\tau_{f,f'} : \mathcal{M}^s_{f,d',\theta'} \to \mathcal{M}^s_{f',d',\theta'}$ in $\mathcal{C}$ such that $[\sigma(h^*(\mathcal{U}_f))] = [(\tau_{f,f'} \circ h)^*(\mathcal{U}_{f'})]$ for every morphism $h : X \to \mathcal{M}^s_{f,d',\theta'}$ in $\mathcal{C}$.*

Proof. We consider the following two functors $F_f, F_{f'} : \mathcal{C} \to \mathrm{Sets}$. For a variety X in $\mathcal{C}$, let $F_f(X)$ (resp. $F_{f'}(X)$) be the set of all S_f-equivalence (resp. $S_{f'}$-equivalence) classes of families of θ'_f-stable (resp. $\theta'_{f'}$-stable) Λ-modules of dimension vector d'_f (resp. $d'_{f'}$). Twisting families by σ induces an isomorphism of functors

$$T_\sigma : F_f \to F_{f'}.$$

Hence $T_\sigma(\mathcal{M}^s_{f,d',\theta'}) : F_f(\mathcal{M}^s_{f,d',\theta'}) \to F_{f'}(\mathcal{M}^s_{f,d',\theta'})$ is bijective. Since F_f (resp. $F_{f'}$) is represented by $\mathcal{M}^s_{f,d',\theta'}$ (resp. $\mathcal{M}^s_{f',d',\theta'}$), T_σ induces a bijective map

$$T_\sigma(\mathcal{M}^s_{f,d',\theta'}) : \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}^s_{f,d',\theta'}, \mathcal{M}^s_{f,d',\theta'}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}^s_{f,d',\theta'}, \mathcal{M}^s_{f',d',\theta'}).$$

Define $\tau_{f,f'}$ to be the image of $\mathrm{id}_{\mathcal{M}^s_{f,d',\theta'}}$ under $T_\sigma(\mathcal{M}^s_{f,d',\theta'})$. Then $[\tau^*_{f,f'}(\mathcal{U}_{f'})] = [\sigma(\mathcal{U}_f)]$.

Suppose now that X is an arbitrary variety in $\mathcal{C}$ and that $h \in \mathrm{Hom}_{\mathcal{C}}(X, \mathcal{M}^s_{f,d',\theta'})$. Then h corresponds to the family $h^*(\mathcal{U}_f)$, and $T_\sigma(X)(h)$ corresponds to the family $\sigma(h^*(\mathcal{U}_f))$. Since $\sigma(h^*(\mathcal{U}_f)) = h^*(\sigma(\mathcal{U}_f))$, we get

$$[\sigma(h^*(\mathcal{U}_f))] = [h^*(\tau^*_{f,f'}(\mathcal{U}_{f'}))] = [(\tau_{f,f'} \circ h)^*(\mathcal{U}_{f'})].$$

Therefore, $T_\sigma(X)(h) = \tau_{f,f'} \circ h$. This means that

$$T_\sigma : F_f = \mathrm{Hom}_{\mathcal{C}}(-, \mathcal{M}^s_{f,d',\theta'}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, \mathcal{M}^s_{f',d',\theta'}) = F_{f'}$$

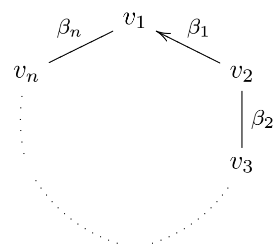
is the isomorphism of functors which corresponds to the composition of morphisms with the morphism $\tau_{f,f'} : \mathcal{M}^s_{f,d',\theta'} \to \mathcal{M}^s_{f',d',\theta'}$. In particular, it follows that $\tau_{f,f'}$ is an isomorphism in $\mathcal{C}$. □

3. Circular modules

Circular quivers have been used by many authors to construct indecomposable modules for algebras (see, for example, [7, 14, 6, 18, 5]). In this section we will show how the methods of the previous section lead to a generalization of some of these results. The modules we construct will be called circular. Our main concern is to find sufficient conditions on the algebra homomorphism $f : \Lambda \to \Lambda'$ so that the hypotheses of Theorem 2.7 are satisfied. This will ensure that the fine moduli spaces $\mathcal{M}^s_{f,d',\theta'}$ of Theorem 2.7 are well-defined. We then apply these results to special biserial algebras Λ, in which case the circular modules are precisely the band modules. We also provide some examples of circular modules for other algebras Λ which are not band modules.

3.1. **Moduli spaces.** As in the previous section, let $\Lambda = KQ/J$ be an arbitrary basic algebra. We first define Λ' and the indecomposable Λ'-modules whose pullbacks will define the circular modules.

Definition 3.1. (i) Let $n \geq 2$, and let Q' be a circular quiver with n vertices $v_1, v_2, \ldots, v_n$ and n arrows $\beta_1, \beta_2, \ldots, \beta_n$ such that Q' contains at least one sink, where we may assume without loss of generality that β_1 points counter-clockwise.

$Q' =$ 

Define Λ' to be the path algebra $\Lambda' = KQ'$.

(ii) We define the word z' corresponding to Q' to be $z' = z_1 z_2 \cdots z_n$ where

$$z_j = \begin{cases} \beta_j & , \quad \beta_j \text{ points counter-clockwise} \\ \beta_j^{-1} & , \quad \beta_j \text{ points clockwise} \end{cases}$$

Let $V_m(\lambda)$ be the indecomposable $K[x, x^{-1}]$-module of K-dimension m such that x acts as the indecomposable Jordan matrix $J_m(\lambda)$, $\lambda \in K^*$. Let $M(z', \lambda, m)$ be the Λ'-module which, as representation of Q', assigns to each vertex the vector space K^m, to the arrow β_1 the map $J_m(\lambda)$, and to all other arrows the identity map. This module is called a band module for Λ' of type z' and is indecomposable (see [5, §3]). There are two cases in which the parameter set K^* of these band modules has to be extended. In case Q' has exactly one sink and z' has the form $z' = \beta_1 \beta_2^{-1} \cdots \beta_n^{-1}$, it makes sense to include $\lambda = 0$ in the parameter set, since $M(z', 0, m)$ behaves like a band module for Λ' (see [4, (2.6)]). In case $n = 2$, Q' has the form $Q' = \cdot \overset{\beta_1}{\underset{\beta_2}{\leftleftarrows}} \cdot$ and it makes sense to include both $\lambda = 0$ and $\lambda = \infty$ in the parameter set, where we define $M(z', \infty, m) = M(\beta_1 \beta_2^{-1}, \infty, m)$ to be $M(\beta_2 \beta_1^{-1}, 0, m)$. Then $M(z', 0, m)$ and $M(z', \infty, m)$ behave like band modules for Λ' (see [4, (2.6)]). Thus the parameter set $K_{z'}$ corresponding to the band modules for Λ' of type z' is either K^*, K, or $K \cup \{\infty\}$. In all cases $K_{z'}$ is an open subset of the projective line $\mathbb{P}^1$ over K, and thus defines a quasi-projective variety over K.

(iii) Let $f : \Lambda \to \Lambda'$ be an algebra homomorphism. Then for each $\lambda \in K_{z'}$ and each $m \in \mathbb{Z}^+$, the pullback $f^* M(z', \lambda, m)$ is called a circular Λ-module.

Remark 3.2. Note that $\Lambda' = KQ'$ is a string algebra. Hence all its indecomposable modules are given as string and band modules, as described e.g. in [5, §3].

We now give a sufficient criterion when the pullbacks of two non-isomorphic band modules for Λ' define two non-isomorphic circular modules for Λ. This will later be used to show that $f : \Lambda \to \Lambda'$ is θ'-stably separated for a certain d' and θ'.

Proposition 3.3. *Suppose Q', Λ', z', and f are as in Definition 3.1. Assume further there exists an algebra automorphism ω of Λ' and a partition $A_1 \cup A_2 \cup \cdots \cup A_r$ of the arrows of Q' such that the following conditions are satisfied:*
Define $\Sigma_i = \sum_{\xi \in A_i} \xi \quad \in \Lambda'$ for $1 \leq i \leq r$.

(i) *The preimage $f^{-1}(\omega(\Sigma_i))$ is non-empty for all $1 \leq i \leq r$.*
(ii) *If for some $1 \leq j \leq n$, $z_j z_{j+1}$ is equal to $\beta_j \beta_{j+1}^{-1}$ or to $\beta_j^{-1}\beta_{j+1}$, then β_j and β_{j+1} do not belong to the same set A_i. Here we use the convention that $z_{n+1} = z_1$.*
(iii) *If ρ is a nontrivial quiver automorphism of Q', then there exists at least one $1 \leq i \leq r$ with $\rho(A_i) \neq A_i$.*

Then the circular module $f^* M(z', \lambda, m)$ is an indecomposable Λ-module for $\lambda \in K_{z'}$ and $m \in \mathbb{Z}^+$. Moreover, $f^* M(z', \lambda, m)$ and $f^* M(z', \mu, l)$ are isomorphic Λ-modules if and only if $\lambda = \mu$ and $m = l$.

Proof. It follows e.g. from [5, p. 161] that $M(z', \lambda, m)$ is an indecomposable Λ'-module for $\lambda \in K_{z'}$ and $m \in \mathbb{Z}^+$, and that $M(z', \lambda, m)$ and $M(z', \mu, l)$ are isomorphic Λ'-modules if and only if $\lambda = \mu$ and $m = l$.

The idea of the proof of Proposition 3.3 is to define a subalgebra Λ_0 of Λ' such that $\Lambda_0 \subseteq f(\Lambda)$ and such that $M(z', \lambda, m)\big|_{\Lambda_0}$ is an indecomposable Λ_0-module for $\lambda \in K_{z'}$ and $m \in \mathbb{Z}^+$, and such that $M(z', \lambda, m)\big|_{\Lambda_0}$ and $M(z', \mu, l)\big|_{\Lambda_0}$ are isomorphic Λ_0-modules if and only if $\lambda = \mu$ and $m = l$. To construct Λ_0 we use results from [11]. Define Q'' to be the quiver with one vertex u and r loop arrows $\xi_1, \xi_2, \ldots, \xi_r$, and define the quiver homomorphism $F : Q' \to Q''$ by $F(v_j) = u$ and $F(\beta_j) = \xi_i$ if $\beta_j \in A_i$. Define I'' to be the ideal in KQ'' generated by all paths of length at least n. Then $\Lambda'' = KQ''/I''$ is a finite dimensional algebra. Moreover, F satisfies the conditions $(W1)$, $(W2)$ and $(W3)$ of [11]. We define an algebra homomorphism $g : KQ'' \to KQ'$ by $g(\xi_i) = \omega(\Sigma_i)$. Then $g(I'') = 0$, since any path in KQ' has length at most $n-1$. Hence g defines an algebra homomorphism $g : \Lambda'' \to \Lambda'$. We define $\Lambda_0 = g(\Lambda'')$. Then Λ_0 is the subalgebra of Λ' generated by $\{\omega(\Sigma_i)\}_{i=1,\ldots,r}$, and therefore $\Lambda_0 \subseteq f(\Lambda)$. It follows as in [11, Remark (1) on p. 188] that $g^* M(z', \lambda, m)$ is an indecomposable Λ''-module for $\lambda \in K_{z'}$ and $m \in \mathbb{Z}^+$. It also follows from the analysis of the maps between tree and band modules in [11] that $g^* M(z', \lambda, m)$ and $g^* M(z', \mu, l)$ are isomorphic Λ''-modules if and only if $\lambda = \mu$ and $m = l$. Since $g(\Lambda'') = \Lambda_0$, we obtain the desired results for the modules $M(z', \lambda, m)\big|_{\Lambda_0}$, $\lambda \in K_{z'}$ and $m \in \mathbb{Z}^+$. This proves Proposition 3.3. $\square$

Remark 3.4. Circular modules are more general than the band modules for arbitrary basic algebras $\Lambda = KQ/J$ as described, for example, in [11]. The algebra homomorphisms $f : \Lambda \to KQ'$ which correspond to such band modules always satisfy conditions (i), (ii) and (iii) of Proposition 3.3. Moreover, for each $1 \leq i \leq r$ there exists a unique arrow α_i in Q with $f^{-1}(\Sigma_i) = \{\alpha_i + J\}$. In §3.3, we will discuss examples of circular modules which are not band modules.

We now define an additive function $\theta' : K_0(\Lambda') \to \mathbb{Z}$ which is preserved by every quiver automorphism of the quiver Q'.

Definition 3.5. For all vertices v_i in Q', let S_i be a simple Λ'-module associated to v_i. We define $\theta'([S_i]) = 1 - b_{i,l} - b_{i,r}$ where $b_{i,l}$ and $b_{i,r}$ are defined as follows.

In case v_i is a source in Q', let $b_{i,l}$ be the number of arrows from v_i to the next sink when walking counter-clockwise around Q', and let $b_{i,r}$ be the number of arrows from v_i to the next sink when walking clockwise.

Suppose now that v_i is not a source in Q'. We have to distinguish between two cases:

(i) The quiver Q' has exactly one source v_{i_0}. Then Q' has exactly one sink v_{i_1}. For $v_i = v_{i_1}$, we let $b_{i,l} = -1$ if $v_{i-1} \neq v_{i_0}$ and $b_{i,l} = 0$ otherwise, and let $b_{i,r} = -1$ if $v_{i+1} \neq v_{i_0}$ and $b_{i,r} = 0$ otherwise. In case $v_i \neq v_{i_1}$, let $b_{i,l} = 1$ if $v_{i-1} = v_{i_1}$ and $b_{i,l} = 0$ otherwise, and let $b_{i,r} = 1$ if $v_{i+1} = v_{i_1}$ and $b_{i,r} = 0$ otherwise. Note that we set $v_{n+1} = v_1$ and $v_0 = v_n$.

(ii) The quiver Q' has at least two sources. Then we let $b_{i,l} = 0 = b_{i,r}$.

Example 3.6. In the following we give a few examples of circular quivers Q' where we label each vertex v_i by $\theta'([S_i])$.

(i) We first look at the case when Q' has exactly one source and one sink.

$$Q' = \begin{tikzcd} 3 & 0 \arrow[l, "\beta_1"'] \\ 0 \arrow[u, "\beta_4"] & -3 \arrow[l, "\beta_3"] \arrow[u, "\beta_2"'] \end{tikzcd} \qquad Q' = \begin{tikzcd} & 2 & & -4 \arrow[ll, "\beta_1"'] \arrow[dr, "\beta_2"] & \\ 0 \arrow[ur, "\beta_5"] & & & & 1 \arrow[dll, "\beta_3"] \\ & & 1 \arrow[ull, "\beta_4"] & & \end{tikzcd}$$

(ii) Now we consider a circular quiver Q' which has 2 sources.

$$Q' = \begin{tikzcd} & 1 & \\ -3 \arrow[ur, "\beta_6"] \arrow[d, "\beta_5"'] & & -1 \arrow[ul, "\beta_1"'] \arrow[d, "\beta_2"] \\ 1 \arrow[dr, "\beta_4"'] & & 1 \\ & 1 \arrow[ur, "\beta_3"'] & \end{tikzcd}$$

Note that in case Q' has at least 2 sources, we get the following result. Suppose v_i and v_j are two sinks when we walk clockwise from v_i to v_j and that there are no other sinks in between v_i and v_j. In this case we will say that v_j is the next clockwise sink from v_j. Let C_{ij} be the string $z_i z_{i+1} \cdots z_{j-1}$, and let $M(C_{ij})$ be the corresponding string module for Λ' (for the definition of string modules, see e.g. [5, §3]). Then $\theta'([M(C_{ij})]) = 1$. In the above example, we get $\theta'([M(\beta_1\beta_2^{-1})]) = 1$ and $\theta'([M(\beta_3\beta_4\beta_5\beta_6^{-1})]) = 1$.

Proposition 3.7. (i) *The additive function $\theta' : K_0(\Lambda') \to \mathbb{Z}$ defined in Definition 3.5 is preserved by every quiver automorphism of Q'.*

(ii) *The isomorphism classes of θ'-stable Λ'-modules of dimension vector $d' = (1,1,\ldots,1)$ are in bijection with the isomorphism classes of the Λ'-band modules $M(z',\lambda,1)$ for $\lambda \in K_{z'}$.*

Proof. Part (i) is clear from the fact that a quiver automorphism of Q' takes sources to sources, sinks to sinks and preserves minimal distances from arbitrary vertices to sinks.

For part (ii) we have to show that all $M(z',\lambda,1)$, $\lambda \in K_{z'}$, are θ'-stable, and that there are no other θ'-stable Λ'-modules of dimension vector $d' = (1,1,\ldots,1)$, up to isomorphism. Let $\lambda \in K_{z'}$. By the definition of θ' it follows that $\theta'([M(z',\lambda,1)]) = 0$. Let now N be a proper indecomposable Λ'-submodule of $M(z',\lambda,1)$. Then it follows that the dimension vector of N has at least one zero entry. Since Λ' is a string algebra, it follows from [5, §3] that N is a string module. Hence $N = M(C')$ for some string C' for Λ'. Since N is a submodule of $M(z',\lambda,1)$, it follows that C' has one of the following forms. The first case is that $C' = 1_{v_i}$ such that v_i is a sink. Then $M(C') \cong S_i$ and $\theta'([M(C')]) = \theta'([S_i]) > 0$. The second case is that C' has length $m \geq 1$ and $C' = z_i z_{i+1} \ldots z_{i+m-1}$ such that z_{i+m} is an arrow in Q' and z_{i-1} is a formal inverse. Here we set $z_{n+1} = z_1$ and $z_0 = z_n$. From Definition 3.5, we have that $\theta'([S_j]) \geq 0$ for all v_j which are not sources. Then C' contains a vertex which is a sink. If Q' has exactly one source, C' cannot contain the source because C' is a proper substring of z'. It follows that $\theta'([M(C')]) > 0$. Suppose Q' has more than one source. It follows from Definition 3.5(ii) and Example 3.6(ii) that $\theta'([S_j]) + \theta'([S_{j+1}]) + \cdots + \theta'([S_k]) = 1$ for all pairs of sinks $v_j \neq v_k$ such that v_k is the next clockwise sink from v_j. Therefore it follows that $\theta'([M(C')]) > 0$, which means that $M(z',\lambda,1)$ is θ'-stable.

Let now M be an arbitrary Λ'-module of dimension vector $d' = (1,1,\ldots,1)$. Then $\theta'([M]) = 0$. Suppose first that M is decomposable. Then $M = M_1 \oplus M_2$, and either $\theta'([M_1]) = 0 = \theta'([M_2])$ which means M is not θ'-stable, or one of $\theta'([M_1])$ or $\theta'([M_2])$ is negative and M is not even θ'-semistable. Suppose now that M is indecomposable of dimension vector d' and not isomorphic to any of $M(z',\lambda,1)$, $\lambda \in K_{z'}$. Then it follows from [5, §3] that M is a string module $M = M(C_i)$ for a string of the form $C_i = z_{i+1} z_{i+2} \cdots z_{i-1}$, $i \in \{1,\ldots,n\}$, where we set $z_{n+1} = z_1$ and $z_0 = z_n$. We need to show that $M(C_i)$ is not θ'-stable. In case Q' has exactly one source and exactly one sink, θ' is given as in Definition 3.5(i). Without loss of generality the sink is v_1 and the source is v_{l+1} for $l \in \{1,\ldots,n-1\}$. Let $k = n - l$. We may assume that $k \geq l$. If $k = l = 1$, the only possible C_i are $C_1 = \beta_2^{-1}$ and $C_2 = \beta_1$. But $M(C_1) \cong M(z',0,1)$ and $M(C_2) \cong M(z',\infty,1)$ are both band modules, so there are no M to consider in this case. If $k > l = 1$, then $\theta'([S_1]) = 2$, $\theta'([S_2]) = -k = -(n-1)$, $\theta'([S_i]) = 1$, $3 \leq i \leq n-1$, and $\theta'([S_n]) = 0$. In this case, $M(C_1) \cong M(z',0,1)$ is a band module, so we do not need to consider it. Since S_n is a proper submodule of $M(C_n)$, $M(C_n)$ is not θ'-stable. For $2 \leq i \leq n-1$, $L_i = M(\beta_1 \beta_2^{-1} \cdots \beta_{i-1}^{-1})$ is a proper submodule of $M(C_i)$ with $\theta'([L_i]) = 2 - k + (i-2) = -(n-1) + i \leq 0$. So $M(C_i)$ is not θ'-stable. If $k \geq l > 1$, then $\theta'([S_1]) = 3$, $\theta'([S_2]) = 0 = \theta'([S_n])$, $\theta'([S_{l+1}]) = -(n-1)$, and for all other j, $\theta'([S_j]) = 1$. In this case, S_2 is a proper submodule of $M(C_1)$, and S_n is a proper submodule of $M(C_n)$. So $M(C_1)$ and $M(C_n)$ are both not θ'-stable. For $2 \leq i \leq l$, $L_i = M(\beta_{i+1} \cdots \beta_l \beta_{l+1}^{-1} \cdots \beta_{n-1}^{-1} \cdots \beta_n^{-1})$ is a proper submodule of $M(C_i)$, and $\theta'([L_i]) = (l-i) - (n-1) + (k-2) + 3 = 2 - i \leq 0$. So $M(C_i)$ is not θ'-stable. Similarly, it follows that $M(C_j)$ is not θ'-stable for $l+1 \leq j \leq n-1$. We now consider $M(C_i)$ in case Q' has at least two sources. Then θ' is given as in Definition 3.5(ii). Let v_h be the next sink when walking counter-clockwise from v_i around Q', and let v_j be the next sink when walking clockwise from v_{i+1} around Q'. Suppose one of the vertices between v_h and v_i, including v_i, is a source, and v_i is not a sink. Then $N_i = M(z_h z_{h+1} \cdots z_{i-1})$ is a proper submodule of $M(C_i)$, since Q' has at least two sources, and $\theta'([N_i]) \leq 0$ (see Example 3.6(ii)). Suppose now one of the vertices between v_{i+1} and v_j, including v_{i+1}, is a source, and v_{i+1} is not a sink. Then $N_i = M(z_{i+1} z_{i+2} \cdots z_{j-1})$ is a proper submodule of $M(C_i)$ and $\theta'([N_i]) \leq 0$ (see Example 3.6(ii)). Since one of these two cases has to occur, it follows that $M(C_i)$ is not θ'-stable. □

The following result establishes a fine moduli space whose points correspond to isomorphism classes of certain circular modules. This theorem is a direct consequence of Theorem 2.7, and Propositions 3.3 and 3.7.

Theorem 3.8. *Suppose Q', Λ', z', and f satisfy the conditions (i), (ii) and (iii) of Proposition 3.3. Suppose further that θ' is as in Definition 3.5 and $d' = (1, 1, \ldots, 1)$. Then f is θ'-stably separated with respect to d', and $\mathcal{M}^s_{f,d',\theta'}$, as defined in Definition 2.5, is a fine moduli space for families of θ'_f-stable Λ-modules of dimension vector d'_f, up to isomorphism. Moreover, the points of $\mathcal{M}^s_{f,d',\theta'}$ are in one-one correspondence with the circular modules $f^* M(z', \lambda, 1)$, $\lambda \in K_{z'}$.*

3.2. **Applications to special biserial algebras.** In this subsection, we describe how Theorems 3.8 and 2.10 can be applied to special biserial algebras.

Definition 3.9. A basic algebra $\Lambda = KQ/J$ is called special biserial, if the following conditions are satisfied.

(i) Any vertex $u \in Q$ is starting point of at most two arrows and end point of at most two arrows.
(ii) For a given arrow $\alpha \in Q$, there is at most one arrow γ such that $\gamma\alpha \notin J$, and there is at most one arrow δ such that $\alpha\delta \notin J$.

If additionally J is generated by paths, Λ is called a string algebra.

Remark 3.10. Suppose Λ is a special biserial algebra. Let $\mathcal{P}$ be a set of submodules of Λ giving a full set of representatives of projective indecomposable Λ-modules which are also injective and not uniserial. Then $\overline{\Lambda} = \Lambda/(\oplus_{P\in\mathcal{P}} \mathrm{soc}(P))$ is a string algebra. Furthermore, the indecomposable $\overline{\Lambda}$-modules are exactly the indecomposable Λ-modules which are not isomorphic to any $P \in \mathcal{P}$.

For string algebras, the indecomposable modules are fully described as string and band modules (see e.g. [5, §3] or [11]). Because of this structure of the indecomposable modules, it follows that special biserial algebras have either finite or tame representation type.

Let $\Lambda = KQ/J$ be a basic special biserial algebra with corresponding string algebra $\overline{\Lambda} = KQ/I$. The band modules for Λ are certain circular modules $f^*\,M(z',\lambda,m)$ where $f : \Lambda \to KQ'$ satisfies conditions (i), (ii) and (iii) of Proposition 3.3. More precisely, f is induced by a quiver homomorphism $F : Q' \to Q$ such that for each $1 \le i \le r$ there exists an arrow α_i in Q with $F(\beta_j) = \alpha_i$ for all $\beta_j \in A_i$. It is also required that for each path v in Q', $F(v)$ does not lie in the ideal I for $\overline{\Lambda}$. Then for each vertex, resp. arrow, a in Q, $f(a+J) = \sum_{\substack{s \text{ in } Q',\\ F(s)=a}} s$, where an empty sum is set equal to zero. In particular, for each $1 \le i \le r$, α_i is the unique arrow in Q with $f(\alpha_i + J) = \Sigma_i$. The word $B = w_1 w_2 \cdots w_n$ in Λ corresponding to $z' = z_1 z_2 \cdots z_n$ and f is defined by

$$w_j = \begin{cases} \alpha_i & , \quad z_j = \beta_j \text{ and } \beta_j \in A_i \\ \alpha_i^{-1} & , \quad z_j = \beta_j^{-1} \text{ and } \beta_j \in A_i \end{cases}$$

and is called a band for Λ. We denote the module $f^*\,M(z',\lambda,m)$ by $M(B,\lambda,m)$. Note that for all $\lambda \in K^*$, $M(B,\lambda,m)$ is indeed a band module for Λ and lies in a 1-tube of the Auslander-Reiten quiver of Λ. But in case the parameter set $K_{z'}$ contains 0 (resp. ∞), $M(B,0,m)$ (resp. $M(B,\infty,m)$) may not behave like a band module, i.e. it may not lie in a 1-tube of the Auslander-Reiten quiver of Λ. For this reason, the parameter set K_B of those values of λ for which $M(B,\lambda,m)$ is a band module for Λ may be properly contained in $K_{z'}$. This happens only if z' has the form $z' = \beta_1\beta_2^{-1}\cdots\beta_n^{-1}$ (see [4, (2.6)] for details on K_B). The modules $M(B,\lambda,m)$, $\lambda \in K_B$, $m \in \mathbb{Z}^+$, are called band modules for Λ of type B.

The next result now follows from Theorem 3.8.

Proposition 3.11. *Let Λ be a basic special biserial algebra, and let $f : \Lambda \to KQ'$ define band modules for Λ of type B. Let $d \in K_0(\Lambda)$ be the dimension vector of the Λ-modules $M(B,\lambda,1)$, $\lambda \in K_B$. Suppose θ', d' and $\mathcal{M}^s_{f,d',\theta'}$ are as in Theorem 3.8 and define $\mathcal{M}_B$ to be the open subvariety of $\mathcal{M}^s_{f,d',\theta'}$ obtained by removing the points corresponding to parameter values in $K_{z'} - K_B$. Then $\mathcal{M}_B$ is a fine moduli space for families of band modules for Λ of type B of dimension vector d, up to isomorphism. The points of $\mathcal{M}_B$ are in one-one correspondence with the band modules $M(B,\lambda,1)$, $\lambda \in K_B$.*

Regarding the action of algebra automorphisms σ of Λ on the moduli spaces $\mathcal{M}_B$, we obtain the following result, which is a consequence of Proposition 3.11 and Theorem 2.10.

Proposition 3.12. *Let σ be an algebra automorphism of Λ such that σ sends band modules of type B to band modules of type B'. Let $\mathcal{U}_B$ (resp. $\mathcal{U}_{B'}$) be the universal family over $\mathcal{M}_B$ (resp. $\mathcal{M}_{B'}$). Then there exists an isomorphism $\tau_{B,B'} : \mathcal{M}_B \to \mathcal{M}_{B'}$ in $\mathcal{C}$ such that $[\sigma(h^*(\mathcal{U}_B))] = [(\tau_{B,B'} \circ h)^*(\mathcal{U}_{B'})]$ for every morphism $h : X \to \mathcal{M}_B$ in $\mathcal{C}$. If $B = B'$, then $\tau_{B,B}$ is an automorphism of $\mathcal{M}_B$. Since $\mathcal{M}_B$ is isomorphic to K_B, $\tau_{B,B}$ is the restriction of an automorphism ρ_B of $\mathbb{P}^1$ to K_B.*

Remark 3.13. Proposition 3.12 essentially reproves [3, Thm. 3.1], and thus [4, Thm. 3.1(ii)], without having to use multi-pushout descriptions for band modules and a lengthy case-by-case analysis.

3.3. **Circular modules which are not band modules.** In this subsection, we give some examples of circular modules which are not band modules in the sense of [11]. We obtain fine moduli spaces for these circular modules from Theorem 3.8 when θ' is as in Definition 3.5 and the dimension vector d' is $(1, 1, \dots, 1)$.

Let first $\Lambda_1 = KQ_1/J_1$ where Q_1 is the quiver

$$Q_1 = \begin{tikzcd} u_0 \arrow[loop left, "\alpha_1"] \arrow[loop right, "\alpha_2"] \arrow[d, shift left, "\alpha_3"] \\ u_1 \arrow[u, shift left, "\alpha_4"] \end{tikzcd}$$

and J_1 is generated by all paths in KQ_1 of length at least 3. Let Q' be the circular quiver

$$Q' = \begin{tikzcd} & v_2 \arrow[dl, "\beta_1"'] & \\ v_1 & & v_3 \arrow[ul, "\beta_2"'] \arrow[dl, "\beta_3"] \\ & v_4 \arrow[ul, "\beta_4"] & \end{tikzcd}$$

and define $f_1 : KQ_1 \to KQ'$ by $f_1(u_0) = v_1 + v_2 + v_3$, $f_1(u_1) = v_4$, $f_1(\alpha_1) = f_1(\alpha_2) = \beta_1 + \beta_2$, $f_1(\alpha_3) = \beta_3$, and $f_1(\alpha_4) = \beta_4$. Then f_1 induces an algebra homomorphism which satisfies $f_1(J_1) = 0$. It follows that f_1 defines an algebra homomorphism $f_1 : \Lambda_1 \to KQ'$. Moreover, f_1 satisfies the conditions (i), (ii) and (iii) of Proposition 3.3. For $z' = \beta_1\beta_2\beta_3^{-1}\beta_4^{-1}$, the circular Λ_1-modules $M_{\lambda,m} = f_1^* M(z', \lambda, m)$, $\lambda \in K_{z'} = K^*$ and $m \in \mathbb{Z}$, are not band modules for Λ_1, which can be seen as follows. The definition of band modules as given in [11] implies that for every band module T for Λ_1 any two different arrows in Q_1 which do not act as zero on T have to act differently on T. Because $f_1(\alpha_1) = f_1(\alpha_2) \neq 0$, the action of α_1 and α_2 is non-zero and identical on $M_{\lambda,m}$ for each $\lambda \in K^*$ and $m \in \mathbb{Z}^+$. Therefore, the modules $M_{\lambda,m}$, $\lambda \in K^*$ and $m \in \mathbb{Z}^+$, give an example of circular modules for Λ_1 which are not band modules.

Let now $\Lambda_2 = KQ_2$ where Q_2 is the quiver

$$Q_2 = \begin{tikzcd} & u_2 \arrow[dl, "\alpha_1"'] & \\ u_1 & & u_3 \arrow[ul, "\alpha_2"'] \arrow[ll, "\alpha_5"'] \arrow[dl, "\alpha_3"] \\ & u_4 \arrow[ul, "\alpha_4"] & \end{tikzcd}$$

Let Q' be the same circular quiver as in the first example, and define $f_2 : KQ_2 \to KQ'$ by $f_2(u_i) = v_i$ for $1 \le i \le 4$, $f_2(\alpha_i) = \beta_i$ for $1 \le i \le 4$, and $f_2(\alpha_5) = \beta_1\beta_2$. Then f_2 defines an algebra homomorphism which satisfies conditions (i), (ii) and (iii) of Proposition 3.3. For $z' = \beta_1\beta_2\beta_3^{-1}\beta_4^{-1}$, the circular Λ_2-modules $N_\lambda = f_2^* M(z', \lambda, 1)$, $\lambda \in K_{z'} = K^*$, are not band modules for Λ_2, which can be seen in the following way. Since the K-dimension of N_λ is 4, any circular quiver S with a winding $F : S \to Q_2$

satisfying $(W1)$, $(W2)$ and $(W3)$ of [11] and giving rise to a band module of K-dimension 4 has at most 4 vertices, hence at most 4 arrows. This means that for any band module for Λ_2 of dimension 4, there is at least one arrow in Q_2 which acts as zero on this module. But no arrow acts as zero on N_λ, $\lambda \in K^*$. Hence the modules N_λ, $\lambda \in K^*$, give another example of circular modules which are not band modules. Note that if $\lambda \neq 1$, the element $\alpha_1\alpha_2 - \alpha_4\alpha_3$ acts nontrivially on N_λ. Thus N_λ is not a module for the quotient $KQ_2/\langle \alpha_1\alpha_2 - \alpha_4\alpha_3\rangle$; modules for this algebra are discussed in [5].

4. Multi-strand modules

In this section we will consider certain representations of quivers which are obtained from the disjoint union of linearly ordered quivers of type A_n for various n, by identifying all sinks to a single vertex and all sources to a single vertex. In other words, these are the same quivers which appear as the Ext quivers of the canonical algebras. However, we will not assume the usual canonical relations for the canonical algebras, but consider instead the path algebras of these quivers.

As in section 2, let $\Lambda = KQ/J$ be an arbitrary basic algebra. We first define Λ' and the indecomposable Λ'-modules whose pullbacks define the multi-strand modules.

Definition 4.1. (i) Let $r \geq 2$ and let $t_1, t_2, \ldots, t_r$ be positive integers. Let $Q' = Q'(t_1, t_2, \ldots, t_r)$ be the following quiver with vertices v_0, v_{r+1}, $v_{i,j}$ and arrows $\alpha_{i,j}$ where $1 \leq i \leq r$, $1 \leq j \leq t_i$:

$$Q' = \begin{tikzcd}
 & & & v_0 \arrow[dlll, "\alpha_{1,1}"'] \arrow[dll, "\alpha_{2,1}"] \arrow[drr, "\alpha_{r,1}"] & & \\
v_{1,1} \arrow[d, "\alpha_{1,2}"'] & v_{1,2} \arrow[d, "\alpha_{2,2}"] & \cdots & & \cdots & v_{r,1} \arrow[d, "\alpha_{r,2}"] \\
v_{1,2} & v_{2,2} & & & & v_{r,2} \\
\vdots & \vdots & & & & \vdots \\
v_{1,t_1-1} \arrow[d, "\alpha_{1,t_1-1}"'] & v_{2,t_2-1} \arrow[d, "\alpha_{2,t_2-1}"] & \cdots & & \cdots & v_{r,t_r-1} \arrow[d, "\alpha_{r,t_r-1}"] \\
v_{1,t_1} \arrow[drrr, "\alpha_{1,t_1}"'] & v_{2,t_2} \arrow[drr, "\alpha_{2,t_2}"] & \cdots & & \cdots & v_{r,t_r} \arrow[dll, "\alpha_{r,t_r}"] \\
 & & & v_{r+1} & &
\end{tikzcd}$$

Define Λ' to be the path algebra $\Lambda' = KQ'$.

(ii) Let $(\ell_1, \ell_2, \ldots, \ell_r) \in K^r - \{(0, 0, \ldots, 0)\}$. Define $M(\ell_1, \ell_2, \ldots, \ell_r)$ to be the Λ'-module given by the following representation

$$\underline{M}(\ell_1, \ell_2, \ldots, \ell_r) = (X_0, X_{i,j}, X_{r+1}, \varphi_{i,j})$$

of Q'. It assigns to v_0 the vector space $X_0 = K$, to $v_{i,j}$ the vector space $X_{i,j} = K$ for $1 \leq i \leq r, 1 \leq j \leq t_i$, and to v_{r+1} the vector space $X_{r+1} = K^r/K \cdot (\ell_1, \ell_2, \dots, \ell_r) \cong K^{r-1}$. For $1 \leq i \leq r$, $1 \leq j \leq t_i - 1$, it assigns to $\alpha_{i,j}$ the identity linear transformation $\varphi_{i,j}$, and to α_{i,t_i} the K-linear map φ_{i,t_i} which sends $1 \in X_{i,t_i} = K$ to the coset $e_i + K \cdot (\ell_1, \ell_2, \dots, \ell_r)$ in $X_{r+1} = K^r/K \cdot (\ell_1, \ell_2, \dots, \ell_r)$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in K^r$ has precisely one non-zero entry at the i-th position. We denote the dimension vector of $M(\ell_1, \ell_2, \dots, \ell_r)$ by $(1, 1, \dots, 1, r-1)$.

(iii) Let $f : \Lambda \to \Lambda'$ be an algebra homomorphism. Then for each $(\ell_1, \ell_2, \dots, \ell_r) \in K^r - \{(0, 0, \dots, 0)\}$, the pullback $f^* M(\ell_1, \ell_2, \dots, \ell_r)$ is called a multi-strand Λ-module.

Lemma 4.2. *Let $Q' = Q'(t_1, t_2, \dots, t_r)$ and $\Lambda' = KQ'$ as in Definition 4.1. Suppose $(\ell_1, \ell_2, \dots, \ell_r), (\ell'_1, \ell'_2, \dots, \ell'_r) \in K^r - \{(0, \dots, 0)\}$. Then $M(\ell_1, \ell_2, \dots, \ell_r)$ is an indecomposable Λ'-module, and $M(\ell_1, \ell_2, \dots, \ell_r) \cong M(\ell'_1, \ell'_2, \dots, \ell'_r)$ if and only if the following two one-dimensional subspaces of K^r are the same:*

$$K \cdot (\ell_1, \ell_2, \dots, \ell_r) = K \cdot (\ell'_1, \ell'_2, \dots, \ell'_r).$$

In other words, $M(\ell_1, \ell_2, \dots, \ell_r) \cong M(\ell'_1, \ell'_2, \dots, \ell'_r)$ if and only if

$$(\ell_1 : \ell_2 : \dots : \ell_r) = (\ell'_1 : \ell'_2 : \dots : \ell'_r)$$

in projective $(r-1)$-space $\mathbb{P}^{r-1}$ over K.

Proof. Since $M(\ell_1, \ell_2, \dots, \ell_r)/\mathrm{rad}(M(\ell_1, \ell_2, \dots, \ell_r))$ is a simple Λ'-module, it follows that $M(\ell_1, \ell_2, \dots, \ell_r)$ is an indecomposable Λ'-module.

Now suppose that the representations of Q' corresponding to the Λ'-modules $M(\ell_1, \ell_2, \dots, \ell_r)$ and $M(\ell'_1, \ell'_2, \dots, \ell'_r)$, as described in Definition 4.1(ii), are isomorphic. For each vertex v in Q', denote the K-linear isomorphism between the vector spaces associated to v by τ_v. Then without loss of generality we can assume that τ_{v_0} is the identity map, which implies that $\tau_{v_{i,j}}$ is the identity map for $1 \leq i \leq r$, $1 \leq j \leq t_i - 1$. Moreover, $\tau_{v_{r+1}}$ has to send $e_i + K \cdot (\ell_1, \ell_2, \dots, \ell_r)$ to $e_i + K \cdot (\ell'_1, \ell'_2, \dots, \ell'_r)$ for all $1 \leq i \leq r$. This implies that

$$K \cdot (\ell_1, \ell_2, \dots, \ell_r) = K \cdot (\ell'_1, \ell'_2, \dots, \ell'_r)$$

which proves Lemma 4.2. □

Remark 4.3. Because of Lemma 4.2, the parameter set corresponding to the isomorphism classes of the Λ'-modules $M(\ell_1, \dots, \ell_r)$ can be identified with projective $(r-1)$-space $\mathbb{P}^{r-1}$.

We now give a sufficient criterion when the pullbacks of two non-isomorphic Λ'-modules of the form $M(\ell_1, \dots, \ell_r)$ and $M(\ell'_1, \dots, \ell'_r)$ define two non-isomorphic multi-strand modules for Λ.

Proposition 4.4. *Suppose Q', Λ', and f are as in Definition 4.1. Define $\Sigma_i = \sum_{j=1}^{t_i} \alpha_{i,j} \in \Lambda'$ for $1 \leq i \leq r$. Suppose that the following condition is satisfied:*

(∗) *The preimage $f^{-1}(\Sigma_i)$ is non-empty for all $1 \leq i \leq r$.*

Then the multi-strand module $f^* M(\ell_1, \ell_2, \ldots, \ell_r)$ is an indecomposable Λ-module for $(\ell_1, \ell_2, \ldots, \ell_r) \in K^r - \{(0, \ldots, 0)\}$. Moreover, two pullbacks $f^* M(\ell_1, \ell_2, \ldots, \ell_r)$ and $f^* M(\ell_1', \ell_2', \ldots, \ell_r')$ are isomorphic Λ-modules if and only if

$$(\ell_1 : \ell_2 : \ldots : \ell_r) = (\ell_1' : \ell_2' : \ldots : \ell_r') \text{ in } \mathbb{P}^{r-1}.$$

Proof. As in the proof of Proposition 3.3, we will define a subalgebra Λ_0 of Λ' such that $\Lambda_0 \subseteq f(\Lambda)$ and such that $M(\ell_1, \ldots, \ell_r)\big|_{\Lambda_0}$ is an indecomposable Λ_0-module for $(\ell_1, \ldots, \ell_r) \in K^r - \{(0, \ldots, 0)\}$, and such that $M(\ell_1, \ldots, \ell_r)\big|_{\Lambda_0}$ and $M(\ell_1', \ldots, \ell_r')\big|_{\Lambda_0}$ are isomorphic Λ_0-modules if and only if $K \cdot (\ell_1, \ldots, \ell_r) = K \cdot (\ell_1', \ldots, \ell_r')$.

Let Λ_0 be the subalgebra of Λ' generated by the identity 1 and by $\{\Sigma_i\}_{i=1,\ldots,r}$. Hence it follows from (*) that $\Lambda_0 \subseteq f(\Lambda)$. Moreover, the Σ_i satisfy the following relations:

$$\begin{aligned} \Sigma_i \cdot \Sigma_j &= 0 \quad \text{for } i \neq j, \\ (\Sigma_i)^{t_i} &= \alpha_{i,t_i} \cdot \cdots \cdot \alpha_{i,2} \cdot \alpha_{i,1}, \\ (\Sigma_i)^{t_i+1} &= 0. \end{aligned}$$

Consider the representation $\underline{M}(\ell_1, \ell_2, \ldots, \ell_r) = (X_0, X_{i,j}, X_{r+1}, \varphi_{i,j})$ of Q' corresponding to the Λ'-module $M(\ell_1, \ell_2, \ldots, \ell_r)$, as described in Definition 4.1(ii). Let $1 \le i \le r$, and let $X_{i,0} = X_0$. Then for $0 \le j \le t_i$, Σ_i acts on $X_{i,j}$ as the linear transformation $\varphi_{i,j}$. This implies that $M(\ell_1, \ldots, \ell_r)\big|_{\Lambda_0}/\mathrm{rad}(M(\ell_1, \ldots, \ell_r)\big|_{\Lambda_0})$ is a simple Λ_0-module, and hence $M(\ell_1, \ldots, \ell_r)\big|_{\Lambda_0}$ is an indecomposable Λ_0-module. Moreover, there is a basis element b_0 of $X_0 = X_{i,0} = K$ such that

$$(\Sigma_i)^{t_i}(b_0) = (\varphi_{i,t_i} \cdot \cdots \cdot \varphi_{i,2} \cdot \varphi_{i,1})\,(b_0) = e_i + K \cdot (\ell_1, \ell_2, \ldots, \ell_r) \in X_{r+1}, \tag{4.1}$$

and hence

$$\left(\ell_1 (\Sigma_1)^{t_1} + \cdots + \ell_r (\Sigma_r)^{t_r}\right)(b_0) = 0.$$

Since $(\Sigma_i)^{t_i}$ acts as the zero linear transformation on $X_{i,j}$ for $1 \le i \le r$, $1 \le j \le t_i$, and $(\Sigma_i)^{t_i}$ also sends $e_i + K \cdot (\ell_1, \ell_2, \ldots, \ell_r)$ to zero for $1 \le i \le r$, this implies that

$$\ell_1 (\Sigma_1)^{t_1} + \cdots + \ell_r (\Sigma_r)^{t_r} = 0. \tag{4.2}$$

Viewing the $(\Sigma_i)^{t_i}$, $1 \le i \le r$, as linear transformations of the K-vector space spanned by basis elements of X_0 and of $X_{i,j}$ for $1 \le i \le r$, $1 \le j \le t_i$, and by $e_i + K \cdot (\ell_1, \ell_2, \ldots, \ell_r) \in X_{r+1}$ for $1 \le i \le r$, we obtain the following: By (4.2) $(\Sigma_1)^{t_1}, \ldots, (\Sigma_r)^{t_r}$ are K-linearly dependent satisfying the particular equation (4.2), and by (4.1) there are $(r-1)$ of them which are linearly independent over K. This implies that $M(\ell_1, \ldots, \ell_r)\big|_{\Lambda_0}$ and $M(\ell_1', \ldots, \ell_r')\big|_{\Lambda_0}$ are isomorphic Λ_0-modules if and only if we have $K \cdot (\ell_1, \ldots, \ell_r) = K \cdot (\ell_1', \ldots, \ell_r')$. □

We now define an additive function $\theta' : K_0(\Lambda') \to \mathbb{Z}$ such that the θ'-stable Λ'-modules of dimension vector $(1, 1, \ldots, 1, r-1)$ are precisely the modules $M(\ell_1, \ldots, \ell_r)$ from Definition 4.1(ii).

Definition 4.5. Let $Q' = Q'(t_1, t_2, \ldots, t_r)$ and $\Lambda' = KQ'$ as in Definition 4.1. We denote by S_0 (resp. $S_{i,j}$ for $1 \le i \le r, 1 \le j \le t_i$, resp. S_{r+1}) a simple Λ'-module corresponding to the vertex v_0 (resp. $v_{i,j}$ for $1 \leq i \leq r, 1 \leq j \leq t_i$, resp. v_{r+1}). We define an additive function $\theta' : K_0(\Lambda') \to \mathbb{Z}$ by

$$\theta'([S_0]) = -(r-1) \cdot (t_1 + \cdots + t_r + 1), \quad \theta'([S_{i,j}]) = (r-1), \quad \theta'([S_{r+1}]) = 1$$

for $1 \leq i \leq r, 1 \leq j \leq t_i$.

It is obvious that θ' is preserved by every quiver automorphism of Q'.

Proposition 4.6. *Let $\theta' : K_0(\Lambda') \to \mathbb{Z}$ be the additive function from Definition 4.5. The isomorphism classes of θ'-stable Λ'-modules of dimension vector $d' = (1, 1, \ldots, 1, r-1)$ are in bijection with the isomorphism classes of the Λ'-modules $M(\ell_1, \ldots, \ell_r)$, as defined in Definition 4.1, for $(\ell_1 : \ldots : \ell_r) \in \mathbb{P}^{r-1}$.*

Proof. We have to show that all $M(\ell_1, \ldots, \ell_r)$, $(\ell_1, \ldots, \ell_r) \in K^r - \{(0, \ldots, 0)\}$, are θ'-stable, and that there are no other θ'-stable Λ'-modules of dimension vector $d' = (1, 1, \ldots, 1, r-1)$, up to isomorphism. Let $(\ell_1, \ldots, \ell_r) \in K^r - \{(0, \ldots, 0)\}$. By the definition of θ' it follows that $\theta'([M(\ell_1, \ldots, \ell_r)]) = 0$. Let now N be a proper Λ'-submodule of $M(\ell_1, \ldots, \ell_r)$. Then it follows that $N \subseteq \mathrm{rad}(M(\ell_1, \ldots, \ell_r))$. Since $M(\ell_1, \ldots, \ell_r)/\mathrm{rad}(M(\ell_1, \ldots, \ell_r))$ corresponds to the vertex v_0 in Q', it follows that $\theta'([N]) > 0$.

Let now M be an arbitrary Λ'-module of dimension vector $d' = (1, 1, \ldots, 1, r-1)$. Then $\theta'([M]) = 0$. Suppose first that M is decomposable. Then $M = M_1 \oplus M_2$, and either $\theta'([M_1]) = 0 = \theta'([M_2])$ which means M is not θ'-stable, or one of $\theta'([M_1])$ or $\theta'([M_2])$ is negative and M is not even θ'-semistable. Suppose now that M is indecomposable of dimension vector d' and denote its corresponding representation of Q' by

$$\underline{M} = (Y_0, Y_{i,j}, Y_{r+1}, \mu_{i,j})$$

where Y_0, $Y_{i,j}$ $(1 \leq i \leq r, 1 \leq j \leq t_i)$ and Y_{r+1} are the vector spaces of dimension 1, 1 and $r-1$, respectively, corresponding to the respective vertices in Q', and $\mu_{i,j}$ $(1 \leq i \leq r, 1 \leq j \leq t_i)$ is the linear transformation corresponding to the arrow $\alpha_{i,j}$ in Q'.

Suppose first that there exist $1 \leq i_0 \leq r, 1 \leq j_0 \leq t_i - 1$ with $\mu_{i_0,j_0} = 0$. Then we obtain a submodule N of M whose corresponding representation $\underline{N} = (Z_0, Z_{i,j}, Z_{r+1}, \nu_{i,j})$ of Q' is obtained from $\underline{M}$ by setting $Z_0 = Y_0$, $Z_{r+1} = Y_{r+1}$, $Z_{i_0,l} = 0$ and $\nu_{i_0,l} = 0$ for $j_0 < l \leq t_{i_0}$, and by setting $Z_{i,j} = Y_{i,j}$ and $\nu_{i,j} = \mu_{i,j}$ for all other $1 \leq i \leq r, 1 \leq j \leq t_i$. Moreover, $\theta'([N]) = -(r-1) \cdot (t_{i_0} - j_0) < 0$, which means that M is not θ'-stable.

Suppose now that $\mu_{i,j} \neq 0$ for all $1 \leq i \leq r, 1 \leq j \leq t_i - 1$. Since M is indecomposable, it follows that the images of μ_{i,t_i}, $1 \leq i \leq r$, have to generate all of $Y_{r+1} \cong K^{r-1}$. But this means that there is an element $(\ell_1, \ldots, \ell_r) \in K^r - \{(0, \ldots, 0)\}$ such that $\underline{M}$ is isomorphic to the representation of Q' obtained from $\underline{M}$ by replacing Y_{r+1} by $K^r / K \cdot (\ell_1, \ldots, \ell_r)$ and μ_{i,t_i} by the linear transformation which sends $1 \in Y_{i,t_i} = K$ to $e_i + K \cdot (\ell_1, \ldots, \ell_r) \in K^r / K \cdot (\ell_1, \ldots, \ell_r)$. Since all the $\mu_{i,j}$, $1 \leq i \leq r, 1 \leq j \leq t_i - 1$, are given by non-zero scalars, it then follows that M is isomorphic to $M\left((\mu_{1,1}^{-1} \cdots \mu_{1,t_1-1}^{-1}) \cdot \ell_1, \ldots, (\mu_{r,1}^{-1} \cdots \mu_{r,t_r-1}^{-1}) \cdot \ell_r\right)$. $\square$

The following result establishes a fine moduli space whose points correspond to isomorphism classes of certain multi-strand modules. This theorem is a direct consequence of Theorem 2.7, and Propositions 4.4 and 4.6.

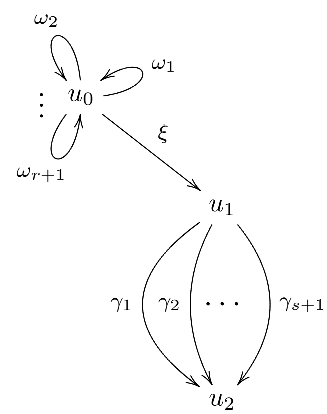
Theorem 4.7. *Suppose Q', Λ', and f satisfy the condition $(*)$ of Proposition 4.4. Suppose further that θ' is as in Definition 4.5 and $d' = (1, 1, \dots, 1, r-1)$. Then f is θ'-stably separated with respect to d', and $\mathcal{M}^s_{f,d',\theta'}$, as defined in Definition 2.5, is a fine moduli space for families of θ'_f-stable Λ-modules of dimension vector d'_f, up to isomorphism. Moreover, the points of $\mathcal{M}^s_{f,d',\theta'}$ are in one-one correspondence with the multi-strand modules $f^*\, M(\ell_1, \dots, \ell_r)$ for $(\ell_1 : \dots : \ell_r) \in \mathbb{P}^{r-1}$.*

5. Example

In this section we will give an example of Λ, Λ', $f : \Lambda \to \Lambda'$, d' and θ' in which the moduli space $\mathcal{M}^s_{f,d',\theta'}$ is isomorphic to $\mathbb{P}^r \times \mathbb{P}^s$. For an appropriate additive function $\theta : K_0(\Lambda) \to \mathbb{Z}$ and dimension vector $d = d'_f \in K_0(\Lambda)$, we will obtain a morphism $\mathcal{M}^s_{f,d',\theta'} \to \mathcal{M}_\Lambda(d,\theta)$ in $\mathcal{C}$ where $\mathcal{M}_\Lambda(d,\theta)$ is King's moduli space and is isomorphic to $\mathbb{P}^s$. This morphism collapses the first factor $\mathbb{P}^r$ of $\mathcal{M}^s_{f,d',\theta'}$.

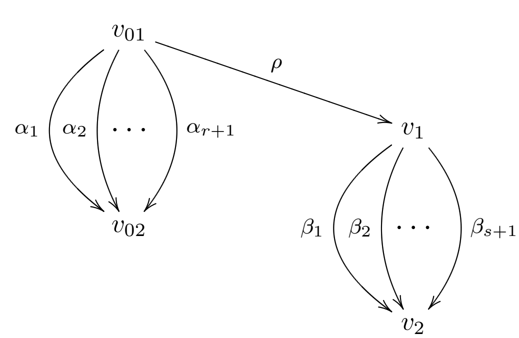
Assume r, s are positive integers. Let $\Lambda = KQ/J$ where Q is the quiver

$Q =$



and $J = \langle \omega_i\omega_j, \xi\omega_j, \omega_i^2 \mid i \neq j, 1 \leq i, j \leq r+1 \rangle$. Let $\Lambda' = KQ'$ where Q' is the quiver

$Q' =$



and define $f : \Lambda \to \Lambda'$ by $f(u_0) = v_{01} + v_{02}$, $f(u_1) = v_1$, $f(u_2) = v_2$, $f(\omega_i) = \alpha_i$, $1 \leq i \leq r+1$, $f(\xi) = \rho$, and $f(\gamma_j) = \beta_j$, $1 \leq j \leq s+1$. Let d' be the dimension vector $d' = (d'_{00}, d'_{01}, d'_1, d'_2) = (1, r, 1, s)$ for Λ', and let $\theta' : K_0(\Lambda') \to \mathbb{Z}$ be the additive function with $\theta'([S_{00}]) = -(r+s+1)$, $\theta'([S_{01}]) = 1$, and $\theta'([S_1]) = 1 = \theta'([S_2])$, where S_* denotes a simple Λ'-module associated to the vertex v_* for $* \in \{01, 02, 1, 2\}$.

The θ'-stable Λ'-modules of dimension vector $d' = (1, r, 1, s)$ can be described as follows.

Let $(\underline{k}, \underline{\ell}) = (k_1, \ldots, k_{r+1}, \ell_1, \ldots, \ell_{s+1}) \in K^{r+1} \times K^{s+1} - \{(\underline{0}, \underline{0})\}$. Define $M(\underline{k}, \underline{\ell})$ to be the Λ'-module given by the following representation

$$\underline{M}(\underline{k}, \underline{\ell}) = (X_{01}, X_{02}, X_1, X_2, \varphi_i, \psi_j)$$

of Q'. It assigns to v_{01} the vector space $X_{01} = K$, to v_{02} the vector space $X_{02} = K^{r+1}/K \cdot (k_1, \ldots, k_{r+1}) \cong K^r$, to v_1 the vector space $X_1 = K$, and to v_2 the vector space $X_2 = K^{s+1}/K \cdot (\ell_1, \ldots, \ell_{s+1}) \cong K^s$. For $1 \le i \le r+1$, it assigns to α_i the K-linear map φ_i which sends $1 \in X_{01} = K$ to the coset $\epsilon_i + K \cdot (k_1, \ldots, k_{r+1})$ in $X_{02} = K^{r+1}/K \cdot (k_1, \ldots, k_{r+1})$, where $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in K^{r+1}$ has precisely one non-zero entry at the i-th position. For $1 \le j \le s+1$, it assigns to β_j the K-linear map ψ_j which sends $1 \in X_1 = K$ to the coset $\eta_j + K \cdot (\ell_1, \ldots, \ell_{s+1})$ in $X_2 = K^{s+1}/K \cdot (\ell_1, \ldots, \ell_{s+1})$, where $\eta_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in K^{s+1}$ has precisely one non-zero entry at the j-th position. Hence the dimension vector of $M(\underline{k}, \underline{\ell})$ is $d' = (1, r, 1, s)$.

Similarly to the proof of Lemma 4.2, one can show that $M(\underline{k}, \underline{\ell}) \cong M(\underline{k}', \underline{\ell}')$ if and only if $(k_1 : \ldots : k_{r+1}) = (k'_1 : \ldots : k'_{r+1})$ in $\mathbb{P}^r$ and $(\ell_1 : \ldots : \ell_{s+1}) = (\ell'_1 : \ldots : \ell'_{s+1})$ in $\mathbb{P}^s$. Moreover, it follows similarly to the proof of Proposition 4.6 that the isomorphism classes of θ'-stable Λ'-modules of dimension vector $d' = (1, r, 1, s)$ are in bijection with the isomorphism classes of the Λ'-modules $M(\underline{k}, \underline{\ell})$ for $(\underline{k}, \underline{\ell}) \in \mathbb{P}^r \times \mathbb{P}^s$.

An argument similar to the proof of Proposition 4.4 shows that the above K-algebra homomorphism $f : \Lambda \to \Lambda'$ is θ'-stably separated with respect to d'. Hence it follows from Theorem 2.7 that $\mathcal{M}^s_{f,d',\theta'}$, as defined in Definition 2.5, is a fine moduli space for families of θ'_f-stable Λ-modules of dimension vector d'_f, up to isomorphism. Moreover, the points of $\mathcal{M}^s_{f,d',\theta'}$ are in one-one correspondence with the modules $f^* M(\underline{k}, \underline{\ell})$ for $(\underline{k}, \underline{\ell}) \in \mathbb{P}^r \times \mathbb{P}^s$.

We now determine which additive functions $\theta : K_0(\Lambda) \to \mathbb{Z}$ can be chosen so that the modules $f^* M(\underline{k}, \underline{\ell})$ for $(\underline{k}, \underline{\ell}) \in \mathbb{P}^r \times \mathbb{P}^s$ are among the θ-semistable Λ-modules of dimension vector $d = d'_f$. Note that $d'_f = (d_0, d_1, d_2) = (r+1, 1, s)$. For $i = 0, 1, 2$, let T_i be a simple Λ-module corresponding to the vertex u_i in Q. We claim that the modules $f^* M(\underline{k}, \underline{\ell})$ for $(\underline{k}, \underline{\ell}) \in \mathbb{P}^r \times \mathbb{P}^s$ are among the θ-semistable Λ-modules of dimension vector $d = d'_f$ only if $\theta([T_0]) = 0$. This can be seen as follows. Each module $f^* M(\underline{k}, \underline{\ell})$ has a direct sum of r copies of T_0 as a submodule of its socle. Hence $\theta([T_0])$ must be non-negative. If $\theta([T_0])$ is strictly positive, say $\theta([T_0]) = a > 0$, then $\theta([T_1]) + s \cdot \theta([T_2]) = -(r+1) \cdot a < 0$. But since $f^* M(\underline{k}, \underline{\ell})$ always has a submodule of dimension vector $[T_1] + s \cdot [T_2]$, this is impossible.

This implies that there are basically two possibilities for θ so that the modules $f^* M(\underline{k}, \underline{\ell})$ for $(\underline{k}, \underline{\ell}) \in \mathbb{P}^r \times \mathbb{P}^s$ are among the θ-semistable Λ-modules of dimension vector $d = d'_f = (r+1, 1, s)$. Namely, either θ is identically zero, or $\theta([T_0]) = 0$, $\theta([T_1]) = -b \cdot s$ and $\theta([T_2]) = b$ for some $0 < b \in \mathbb{Z}$, in which case we can choose without loss of generality $b = 1$.

In case θ is identically zero, all Λ-modules are θ-semistable, the only θ-stable Λ-modules are the simple Λ-modules, and all Λ-modules of a given dimension vector are S-equivalent. Hence King's moduli space $\mathcal{M}_\Lambda(d,\theta)$ is a point.

In case $\theta([T_0]) = 0$, $\theta([T_1]) = -s$ and $\theta([T_2]) = 1$, we claim that a Λ-module N of dimension vector $d = d'_f = (r+1, 1, s)$ is θ-semistable if and only if the corresponding representation $\underline{N}$ of the quiver Q assigns to each arrow γ_j, $1 \le j \le s+1$, a K-linear transformation $\Gamma_j : K \to K^s$ such that the images of Γ_j, $1 \le j \le s+1$, generate all of K^s. If this were not true, one could write N as a direct sum $N_1 \oplus N_2$ in which N_1 is a complement in K^s to the span of the images of the Γ_j and $\theta([N_2]) < 0$, contradicting θ-semistability. In particular, every θ-semistable N of dimension vector d has a θ-stable proper submodule U_N which results from changing $\underline{N}$ by assigning to u_0 the zero vector space and to ξ and ω_i, $1 \le i \le r+1$, the zero linear transformations. By Definition 2.1(ii), this implies that two θ-semistable Λ-modules N and N' of dimension vector $d = d'_f = (r+1, 1, s)$ are S-equivalent if and only if the corresponding submodules U_N and $U_{N'}$ are isomorphic. Hence we can describe the S-equivalence classes of θ-semistable Λ-modules of dimension vector d as follows. For every fixed choice of the first parameter $\underline{k}_0 \in K^{r+1}$ involved in defining the Λ'-modules $M(\underline{k}_0, \underline{\ell})$, the S-equivalence classes of θ-semistable Λ-modules of dimension vector d are in one-one correspondence with the Λ-modules $f^* M(\underline{k}_0, \underline{\ell})$ for $\underline{\ell} \in \mathbb{P}^s$. This means that for this θ, King's moduli space $\mathcal{M}_\Lambda(d,\theta)$ is isomorphic to $\mathbb{P}^s$. By considering functors as in Remark 2.8, we obtain a morphism

$$\mathcal{M}^s_{f,d',\theta'} \cong \mathbb{P}^r \times \mathbb{P}^s \to \mathcal{M}_\Lambda(d,\theta) \cong \mathbb{P}^s$$

in $\mathcal{C}$ which collapses the first factor $\mathbb{P}^r$ of $\mathcal{M}^s_{f,d',\theta'}$.

References

[1] M. Auslander, I. Reiten, S. O. Smalø, Representation theory of Artin algebras, Cambridge Studies in Adv. Math. 36, Cambridge University Press, Cambridge, 1995.

[2] F. M. Bleher, Automorphisms of string algebras, J. Algebra 201 (1998), 528–546.

[3] F. M. Bleher, Operations of automorphisms on 1-tubes, J. Pure Appl. Algebra 138 (1999), 1–20.

[4] F. M. Bleher, Special biserial algebras and their automorphisms, Contemp. Math. 229 (1998), 17–27.

[5] M. C. R. Butler and C. M. Ringel, Auslander-Reiten-Sequences with few middle terms and applications to string algebras, Comm. in Alg. 15 (1987), 145–179.

[6] P. W. Donovan, M. R. Freislich, Indecomposable representations of certain groups with dihedral Sylow subgroups, Math. Ann. 238 (1978), 207–216.

[7] I. M. Gelfand, V. A. Ponomarev, Indecomposable representations of the Lorentz group, Russian Math. Surveys 23 (1968), 1–58.

[8] B. Huisgen-Zimmermann, Classifying representations by way of Grassmannians, Trans. Amer. Math. Soc. 359 (2007), 2687–2719.

[9] B. Huisgen-Zimmermann, Top-stable degenerations of finite dimensional representations I, Preprint.

[10] A. D. King, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford Series (2) 45 (1994), 515–530.

[11] H. Krause, Maps between tree and band modules, J. Algebra 137 (1991), 186–194.

[12] H. Nakajima, Varieties associated with quivers, in: Representation theory of algebras and related topics (Mexico City, 1994), pp. 139–157, CMS Conf. Proc., 19, Amer. Math. Soc., Providence, RI, 1996.

[13] P. E. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute of Fundamental Research, Bombay, published by Springer-Verlag, Berlin-New York, 1978.

[14] C. M. Ringel, The indecomposable representations of dihedral 2-groups, Math. Ann. 214 (1975), 19–34.

[15] A. Schofield, Birational classifications of moduli spaces, in: Infinite length modules (Bielefeld, 1998), pp. 297–309, Trends Math., Birkhäuser, Basel, 2000.

[16] C. S. Seshadri, Space of unitary vector bundles on a compact Riemann surface, Ann. of Math. 85 (1967), 303–336.

[17] J. P. Serre, Faisceaux algébriques cohérents. Ann. of Math. (2) 61 (1955), 197–278.

[18] B. Wald and J. Waschbüsch, Tame biserial algebras, J. Algebra 95 (1985), 480–500.

F.B.: Department of Mathematics, University of Iowa, Iowa City, IA 52242-1419
E-mail address: `fbleher@math.uiowa.edu`

T.C.: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395
E-mail address: `ted@math.upenn.edu`