A Note on an Identity of Feigin-Stoyanovsky

Herbert S. Wilf
Department of Mathematics, University of Pennsylvania
Philadelphia, PA 19104-6395, USA
<wilf@math.upenn.edu>

December 23, 1998

We use the standard notation of hypergeometric series by defining

$$(a;q)_k = \begin{cases} \left((1-aq^{-1})(1-aq^{-2})...(1-aq^{k})\right)^{-1}, & \text{if } k<0; \\ 1, & \text{if } k=0; \\ (1-a)(1-aq)\dots(1-aq^{k-1}), & \text{if } k>0. \end{cases}$$

Then the identity of the title, which was communicated to me by Alexandre Kirillov, is

$$\sum_{k=0}^{m} \frac{(q^{k+1};q)_{-k}(q^{m-k+1};q)_{k-m}(bq^{k+1};q)_{k-m}(q^{m-k+1}/b;q)_{-k}}{(q^{m+1};q)_{-m}(abq;q)_{k-m}(cq;q)_{k-m}(cq/b;q)_{-k}} = \frac{1}{(acq;q)_{-m}}. \tag{1}$$

1 The computer proof via the q-WZ method

Following the WZ paradigm, we define $F(m,k)$ to be the summand in (1) divided by the right hand side, i.e.,

$$F(m,k) = \frac{(q^{k+1};q)_{-k}(q^{m-k+1};q)_{k-m}(bq^{k+1};q)_{k-m}(q^{m-k+1}/b;q)_{-k}(acq;q)_{-m}}{(q^{m+1};q)_{-m}(abq;q)_{k-m}(cq;q)_{k-m}(cq/b;q)_{-k}}, \tag{2}$$

and now what we have to prove is that

$$f(m) \stackrel{\text{def}}{=} \sum_k F(m,k) = 1 \qquad (m \ge 0). \tag{3}$$

In order to do this it will suffice to exhibit a function $G(m,k)$ such that

$$F(m,k) - F(m-1,k) = G(m,k) - G(m,k-1), \tag{4}$$

and such that $G(m,k)$ vanishes, for each fixed m, as $k \to \pm\infty$.

For suppose we have found such a G. Then sum both sides of (4) over all integers k, observe that the right side telescopes to zero, and obtain the result that

$$f(m) - f(m-1) = 0 \qquad (m \ge 0),$$

where f is defined in (3) above. But this clearly implies that f is constant on the nonnegative integers, and since $f(0)=1$, we will have shown the desired result (3).

A function G which satisfies the above properties is given by

$$G(m,k)=R(q^m,q^k)F(m,k) \tag{5}$$

where $F(m,k)$ is given by (2) and the rational function R is

$$R(x,y)=\frac{qx(y-a)(by-c)(y-x)(by-x)}{(x-by^2)(x-1)(x-abqy)(x-cqy)}. \tag{6}$$

To prove that the WZ equation (4) is satisfied, one can divide it by $F(m,k)$, and show that

$$1-\frac{F(m-1,k)}{F(m,k)}=R(q^m,q^k)-R(q^m,q^{k-1})\frac{F(m,k-1)}{F(m,k)}. \tag{7}$$

Both sides of this equation are *rational functions* of $x=q^m, y=q^k$, so the verification is straightforward, by clearing of fractions and checking the resulting polynomial identity.

To make this note self-contained, so the reader can readily check the entire computation we will exhibit the left and right ratios $F(m-1,k)/F(m,k)$ and $F(m,k-1)/F(m,k)$. These are obtained directly from the definition (2) of F, and they are

$$\frac{F(m-1,k)}{F(m,k)} = \frac{(x-acq)\ (y-x)\ (by-x)\ \left(x-b\,q\,y^2\right)}{(1-x)\ (x-a\,b\,q\,y)\ (x-c\,q\,y)\ (b\,y^2-x)} \qquad (x=q^m;\ y=q^k) \tag{8}$$

$$\frac{F(m,k-1)}{F(m,k)} = \frac{q\ (1-y)\ (1-b\,y)\ (x-a\,b\,y)\ (x-c\,y)\ \left(b\,y^2-qx\right)\ \left(b\,y^2-q^2x\right)}{(y-aq)\ (y-qx)\ (b\,y-cq)\ (b\,y-qx)\ (x-b\,y^2)\ (q\,x-b\,y^2)} \qquad (x=q^m;\ y=q^k) \tag{9}$$

A complete verification of this proof now proceeds as follows.

1. Check that the left and right ratios of the summand F are as shown above in (8) and (9).
2. Substitute these in (7), using the definition (6) of the rational function R, and thereby verify that (7) is correct.
3. The truth of the WZ equation (4) is now established. The remainder of the argument is as given previously.

2 Comments

The computerized portion of the proof takes as input the summand $F(m,k)$ of (3), and finds the rational function R, of (6), such that if G is defined by (5), then the WZ equation (4) is satisfied. There are several software implementations of the q-Zeilberger algorithm that can routinely accomplish this task. The one we used is the one due to Axel Riese, of RISC-Linz, and which can be found at

< http://www.risc.uni-linz.ac.at/research/combinat/risc/software/qZeil/ >.