Preface

We now know ([Z1], [WZ1]) that a large class of special function identities and binomial coefficient identities, are verifiable in a *finite* number of steps, since they can be embedded in the class of *holonomic function identities*, the elements of which are specifiable by a *finite* amount of data. Alas, the *finite* is usually a very big finite, and for most identities the holonomic approach [Z1], that uses an elimination algorithm in the Weyl algebra, is only of theoretical interest. In [Z2], [Z3], [WZ1], and [WZ2], it was shown that for *single sum* terminating hypergeometric identities (i.e. binomial) and transformation formulas there exists a *fast* algorithm producing one- or two-line elegant proofs (called "certificates") for any desired identity. Very often [WZ2] one also obtains, as a bonus, new identities, called the *dual* and the *companion*.

In this paper we show that these fast algorithms can be extended to the much larger class of *multi-sum* terminating hypergeometric (or equivalently, binomial coefficient) identities, to constant term identities of Dyson-Macdonald type, to Mehta-Dyson type integrals, and more generally, to identities involving any (fixed) number of sums and integrals of products of *special functions of hypergeometric type*. The computer-generated proofs obtained by our algorithms are always short, are often very elegant, and like the single-sum case, sometimes yield the discovery and proof of new identities. We also do the same for single- and multi- (terminating) q-hypergeometric identities, with continuous and/or discrete variables.

Here we describe these algorithms in general, and prove their validity. The validity is an immediate consequence of what we call "The fundamental theorem of hypergeometric summation and integration", a result which we believe is of independent theoretical interest and beauty. The technical aspects of our algorithms, as well as their implementation in Maple, will be described in a forthcoming paper.

It is possible, and sometimes preferable, to enjoy a magic show without understanding how the tricks are performed. Hence we invite casual readers to go directly to section 6, in which we give several examples of one- or two- line proofs generated by our method. In order to understand these proofs, and convince oneself of their correctness, one doesn't need to know how they were generated. Readers can generate many more examples on their own once they obtain a copy of our Maple program, that is available upon request from `<zeilberg@euclid.math.temple.edu>`.

The present work generalizes from *one* to *many* variables, our previous work ([Z2], [Z3], [WZ1], [WZ2], [AZ]), i.e. from one $\sum$ or $\int$ to several such. However, it is not possible yet to generalize from *specifically many* to *arbitrarily many*, by purely computerized methods. For example Macdonald's [Ma] constant term conjectures for root systems, for a *specific* root system, is doable by the present method, but we can not yet do it for *all* root systems at once. However, as our computers keep getting better and better, we can use them to generate proofs of special cases of multivariate identities, for small numbers of variables, or root systems. Then a human might detect a common pattern that can be generalized to give a "one-line",[1] though human, proof of the general identity, valid for an *arbitrary* number of $\sum$'s and variables. Examples of this process are given in section 6.5, where we give a new, extremely short and elementary, proof of Holman's [Ho] $U(N)$ generalization of Gauss's hypergeometric identity, and a new proof of Selberg's ([Se][An1]) celebrated generalization of Euler's beta-integral. These proofs would not have been possible (at least for us), without the computer-generated WZ proofs given for the cases $N = 1, 2, 3$, which possessed a clear, easily generalizable, pattern. We feel that such uses of our algorithms are the most interesting and

[1] The reason we put *one line* in quotes, is that unlike identities with *fixed* numbers of sigmas and integrals, in which the *one line* is a purely routine finite algebraic identity, the "one line" for multivariate identities for a general number of sigmas and/or integral signs, is not purely routine (at present) and requires a "human" proof, though probably a very simple one.

humanly rewarding, and will make them a useful research tool in the still young subject of *multivariate* identities.

We illustrate the stage of development of our research program by the trivial example of the binomial theorem. For the Renaissance mathematician, for a junior high school student, and for a Fortran programmer the formulas

$$(x+y)^2 = x^2 + 2xy + y^2, \quad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

are "interesting" general facts, since they contain an *infinite* number of numerical facts, obtained by plugging in specific values for x and y. However, to a senior high school student, a 17th century mathematician, or a Maple programmer, these are purely routine *single* facts, since x and y are considered as indeterminates. One can then use Maple to expand $(x+y)^n$, for $n = 1, 2, 3, 4, \ldots$, and conjecture empirically the general *binomial theorem*

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

which before the appearance of [Z1], [Z2], [WZ1], [WZ2] was considered a "human" theorem. Thanks to these papers, this is now purely routine, since it can be proved by computer. The present paper makes it possible to prove the *trinomial*, and more generally, the *multinomial* theorem for any *fixed* number (in practice < 10) of variables. However we still need a human to do this for the *general* multinomial theorem involving an arbitrary number of variables.

Our theory is concerned directly only with *terminating* identities, but many non-terminating identities (Koornwinder [K] conjectures *all*) are immediate (human) consequences of terminating ones, usually with extra parameters. For example, the Rogers-Ramanujan identities themselves are not computer provable (by our algorithms), but their "finite form" extensions are [ET]. Hence the present theory applies, albeit indirectly, to non-terminating identities.

We will now describe the contents of the paper. Section 1 presents the historical and mathematical context of our work, and introduces the notion of hypergeometric terms. Section 2 presents our centerpiece result, the *fundamental theorem of hypergeometric summation and integration*, in the full generality of functions of several discrete and continuous variables, as well as its *q-analog*. They follow almost immediately from our *fundamental lemma*, and its q-analog, respectively. Our proof for the ordinary case uses *holonomic systems* [Be], [Bj], as applied in [Z1]. This proof has the drawback, however, that although it guarantees the *existence* of the things promised by the fundamental theorem, it doesn't produce effective upper bounds for the orders of the recurrences and differential equations involved. That is why, in sections 3 and 4, we present an alternative approach in the case of discrete hypergeometric functions, that yields explicit upper bounds, and furthermore is entirely elementary. This method, based on the pioneering work of Sister Celine Fasenmyer [Fa] (see also [Ve], [Z0]), also has the advantage that it generalizes naturally to the proof of the *q-fundamental lemma*, which is carried out in section 5. Luckily, in this case, no extra effort is required for the continuous case, since, as we will show, it's completely "isomorphic". The final section contains many examples that, as pointed out above, can be read and understood independently of the rest of the paper.

A few words about nomenclature. Throughout this paper, $(a)_n$ denotes the *rising factorial* $a(a+1)\ldots(a+n-1)$, whenever it appears in an *ordinary* hypergeometric series. When it appears in a q-series, however, it means $(a;q)_n := (1-a)(1-qa)\ldots(1-q^{n-1}a)$.

1. Introduction

1.1 Special functions of mathematical physics.

Laplace and his contemporaries believed that the universe is governed *exactly* by differential equations, and, that given the initial conditions, one would be able to predict the future for ever after. That is why late-18th and 19th century mathematicians (most of whom were also physicists) were occupied in trying to solve *explicitly* Laplace's equation, the wave equation, and other partial differential equations, under various symmetries. By separation of variables, one was able to go from one partial differential equation to several ordinary differential equations, and thus were born the *special functions of mathematical physics*: the functions and polynomials bearing the names of Legendre, Hermite, Laguerre, Jacobi, Bessel, etc. When Laplace's world-view was shattered by Heisenberg and his school, these special functions did not become *passé*, but on the contrary, adapted themselves to serve the new reigning quantum queen very much as Laplace himself adapted, and managed always to save his head during the political changes in his lifetime. The reason was, of course, that the wave equation of classical physics became Schrödinger's equation of quantum physics, and so the loot of special functions was used with a vengeance. In our own century, new special functions were found, and some of them, like the Krawchouk polynomials, turned out to be useful in that by-product of twentieth century electronic communication called coding theory, and other branches of discrete mathematics.

The so-called special functions of mathematical physics turn out to have beautiful and *useful* properties. Perhaps the most striking feature is that very often they are *orthogonal* with respect to an appropriate weight (measure). This implies nice properties of their zeroes, and so we can use them for *numerical quadrature*, as was shown by Gauss. Another surprising property, that is not unrelated to the foregoing, is that they are usually expressible as *hypergeometric series*.

Recall that a series $\sum_{k=0}^{\infty} a_k$ is *hypergeometric* if the ratio of consecutive terms a_{k+1}/a_k is a *rational function* of k. This property leads to a systematized and unified notation and *theory* for special functions. However, a much stronger property, that must have been used implicitly many times, but that was not distilled and pointed out until recently ([Z0], [Z1], [WZ2]) holds in the vast majority of cases. Most special functions can be written as

$$P_n = \sum_{k=0}^{\infty} F(n,k),$$

where n is an auxiliary parameter, and one has that not only is $F(n,k+1)/F(n,k)$ a rational function in k, but is a rational function in (k,n), and in addition, so is $F(n+1,k)/F(n,k)$. We will call such an F a *hypergeometric term*. This makes the object of interest an entirely rational, finitary object, and raises the possibility, already realized in [Z1], [WZ1], [WZ2], that it can be handled by finite methods and machines.

We pause for a moment to look at some venerable special functions.

For the Hermite polynomials

$$H_n(x) := n! \sum_k \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!}, \tag{Hermite}$$

we have $F(n+1,k)/F(n,k) = 2x(n+1)/(n-2k-1)$, and $F(n,k+1)/F(n,k) = -(n-2k)(n-2k-1)/(4x^2(k+1))$.

The Laguerre polynomials

$$L_n^{\alpha}(x) := \sum_k \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \tag{Laguerre}$$

have $F(n+1,k)/F(n,k) = (n+\alpha+1)/(n-k+1)$, and $F(n,k+1)/F(n,k) = (n-k)(-x)/((\alpha+k+1)(k+1))$.

In the Legendre case, we have

$$P_n(x) := \frac{1}{2^n}\sum_k \binom{n}{k}^2 (x-1)^k (x+1)^{n-k}, \qquad \text{(Legendre)}$$

and then $F(n+1,k)/F(n,k) = ((x+1)/2)(n+1)^2/(n-k+1)^2$ and $F(n,k+1)/F(n,k) = (n-k)^2(x-1)/((x+1)(k+1)^2)$.

For the general Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) := \frac{1}{2^n}\sum_k \binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}, \qquad \text{(Jacobi)}$$

we find that

$$\frac{F(n,k+1)}{F(n,k)} = \frac{(x+1)(n+\alpha+1)(n+\beta+1)}{(n-k+1)(n+\beta-k+1)},$$

and

$$\frac{F(n+1,k)}{F(n,k)} = \frac{(x-1)(n-k)(n+\beta-k)}{(\alpha+k+1)(k+1)(x+1)}.$$

We notice that in addition to the distinguished discrete parameter n, there are other parameters/variables. First there is the *continuous variable* x, and in case of the Laguerre polynomials we have another parameter α, while in the case of the Jacobi polynomials we have two extra parameters α and β. It turns out that whenever that is the case, it is also true that F is hypergeometric in *all* of its variables and parameters. In other words, the ratios $F(\alpha+1)/F(\alpha)$, and $F(\beta+1)/F(\beta)$ are rational functions in all of the variables. Indeed, calling the summand of (Jacobi) $F(n,k,\alpha,\beta,x)$, we have that

$$\frac{F(n+1,k,\alpha,\beta,x)}{F(n,k,\alpha,\beta,x)}, \quad \frac{F(n,k+1,\alpha,\beta,x)}{F(n,k,\alpha,\beta,x)}, \quad \frac{F(n,k,\alpha+1,\beta,x)}{F(n,k,\alpha,\beta,x)},$$
$$\frac{F(n,k,\alpha,\beta+1,x)}{F(n,k,\alpha,\beta,x)}, \quad \frac{1}{F(n,k,\alpha,\beta,x)}\frac{\partial F(n,k,\alpha,\beta,x)}{\partial x},$$

are all *rational functions* of *all* of the arguments (n,k,α,β,x).

Special functions and hypergeometric series satisfy many identities. For example[2] (see [As], p. 39),

$$\int_{-1}^{1} P_m(x)P_n(x)P_{m+n-2k}(x)dx = \frac{1}{(m+n+1/2-k)}\frac{A_k A_{m-k}A_{n-k}}{A_{m+n-k}}, \qquad \text{(Adams)}$$

where $A_k := (1/2)_k/k!$. Another typical formula is the following, which received a beautiful combinatorial proof, and was generalized to several variables, by Foata and Strehl [FS]:

$$\sum_{n=0}^{\infty} \frac{u^n n!}{(\alpha+1)_n} L_n^{(\alpha)}(x)L_n^{(\alpha)}(y) = (1-u)^{(-\alpha-1)}\exp\left\{\frac{-(x+y)u}{(1-u)}\right\}\sum_n \frac{1}{n!(\alpha+1)_n}\left(\frac{xyu}{(1-u)^2}\right)^n. \qquad \text{(Hille-Hardy)}$$

Readers are encouraged to browse through [R] and [Er] for literally hundreds of other formulas involving special functions. In [Z1] it was shown that all such formulas, involving summations and integrations of

[2] This formula is the sister of a planet: Adams also discovered Neptune!

products of special functions of hypergeometric type are verifiable in a *finite* number of steps. This was done by noting that since the summand is always *proper hypergeometric* (see below), it is in particular *holonomic*, and hence, the sum itself is holonomic. Since a holonomic function is specifiable by a finite amount of information, in a uniform way, and it is possible to find representations for products, sums, and integrals, it followed that all such identities were decidable in a *finite* number of steps.

Alas, what is true in principle turns out to be very time consuming in practice. In [Z2], [Z3], [WZ1], and [WZ2] it was noticed that for *single* hypergeometric sums, and for *single* so-called hyperexponential integrals [AZ], there exist extremely fast algorithms, that exploit fully the fact that the summand is *hypergeometric*, as opposed to the holonomic approach, that uses only the fact that the summand is holonomic. These algorithms used an extension of Gosper's [Gos] algorithm for single-sum *indefinite* hypergeometric summation, and of its continuous analog.

We will show that these fast algorithms of [Z2], [WZ1], [WZ2], and [AZ] extend naturally to the most general situation of multisums/integrals of special functions of hypergeometric type. The key observation is that *every expression involving definite sums and integrals of special functions of hypergeometric type can be rewritten as another expression, usually with more sigmas and integral signs, but in which the integrand-summand is a hypergeometric term.*

For example the fully expanded version of (Adams) obtained by plugging (Legendre) into it is

$$\int_{-1}^{1} dx \sum_{s_1}\sum_{s_2}\sum_{s_3} \binom{m}{s_1}^2 \binom{n}{s_2}^2 \binom{m+n-2k}{s_3}^2 (x-1)^{s_1+s_2+s_3}(x+1)^{2m+2n-2k-s_1-s_2-s_3}$$
$$= \frac{1}{(m+n+1/2-k)} \frac{A_k A_{m-k} A_{n-k}}{A_{m+n-k}}. \tag{Adams-expanded}$$

Hence Adams's formula has the form

$$\int_{-1}^{1} dx \sum_{s_1}\sum_{s_2}\sum_{s_3} F(m,n,k,s_1,s_2,s_3,x) = G(m,n,k),$$

where F is hypergeometric in discrete variables m, n, k, s_1, s_2, s_3 and the continuous variable x, and $G(m,n,k)$ is hypergeometric in discrete variables m, n, and k. Similarly, the spelled out Hille-Hardy formula, reads as

$$\sum_{n=0}^{\infty}\sum_{k_1}\sum_{k_2} \frac{u^n n!}{(\alpha+1)_n} \binom{n+\alpha}{n-k_1} \frac{(-x)^{k_1}}{k_1!} \binom{n+\alpha}{n-k_2} \frac{(-y)^{k_2}}{k_2!} =$$
$$(1-u)^{-\alpha-1} \exp\{-\frac{(x+y)u}{(1-u)}\} \sum_n \frac{1}{n!(\alpha+1)_n} \left(\frac{xyu}{(1-u)^2}\right)^n . \tag{Hille-Hardy-expanded}$$

The format of the Hille-Hardy identity is thus

$$\sum_n \sum_{k_1}\sum_{k_2} F(n,k_1,k_2,x,y,u) = \sum_n G(n,x,y,u),$$

where F and G are hypergeometric functions of their arguments.

The two examples above assert the equality of two expressions of the kind

$$\sum_{\mathbf{k}} \int F(\mathbf{n},\mathbf{k},\mathbf{x},\mathbf{y}) d\mathbf{y}, \tag{general-integral-sum}$$

where F is a hypergeometric term. Such expressions, and identities between them, are the "objects" of study in our theory. In the above formula, $\mathbf{k}$, $\mathbf{n}$, are discrete multi-variables, while $\mathbf{x}$ and $\mathbf{y}$, are continuous multi-variables, and F is *hypergeometric* in all of its arguments. Sometimes, as in the right side of Adams's formula, $\mathbf{k}$ and $\mathbf{y}$ are empty, so the right side is already hypergeometric. We will present a general algorithm that produces a one- or two-line proof of any such identity.

Two special cases of the general (integral-sum) deserve special mention. One is the case of the pure *multi-sum* (i.e. $\mathbf{y}$ is empty), in which we get a *combinatorial sum*, and the other is the case of the pure *multi-integral*, (i.e. $\mathbf{k}$ is empty), which includes Dyson-Macdonald constant term expressions, Selberg's integral, and Mehta-Dyson-Macdonald integrals. Let's say a little more about these.

1.2 Combinatorial sums.

In enumerative combinatorics or discrete probability, one attempts to find expressions $a(\mathbf{n})$ enumerating families of finite sets described by certain conditions that are parametrized by $\mathbf{n}$. Many times, complex combinatorial sets can be expressed as (disjoint) unions and cartesian products of "basic events", that very often turn out to be acts of "choosing". Hence, the importance in combinatorics of the binomial coefficients $\binom{n}{k} = n!/(k!(n-k)!)$, pronounced *n choose k*, the number of ways of choosing k objects out of n objects, and more generally, the *multinomial coefficients*

$$\binom{m_1 + \cdots + m_r}{m_1, \ldots, m_r} = \frac{(m_1 + \cdots + m_r)!}{m_1! \ldots m_r!}, \qquad \text{(multinomial)}$$

which are the number of ways of choosing m_1 objects to do one thing, m_2 objects to do a second thing, $\ldots$, m_r objects to do an rth thing, out of a total of $m_1 + \cdots + m_r$ objects. The operation of cartesian product turns, upon counting, into multiplication, and that of disjoint union, into addition. Thus, we often get *sums of products of binomial and multinomial coefficients*, all of which fall under the present heading. For example, if on each day of Christmastide one tosses a fair coin n times, the probability that one gets the same number of heads each day is

$$C(n) = \frac{1}{2^{12n}} \sum_{k=0}^{n} \binom{n}{k}^{12}.$$

The Jewish analog of this is

$$\text{ח}(n) = \frac{1}{4^{8n}} \sum_{\alpha_{\text{נ}} + \alpha_{\text{ה}} + \alpha_{\text{ג}} + \alpha_{\text{ש}} = n} \binom{n}{\alpha_{\text{נ}}, \alpha_{\text{ה}}, \alpha_{\text{ג}}, \alpha_{\text{ש}}}^8.$$

1.3 Multi-integrals and constant term expressions.

Dyson's erstwhile conjecture (see [Goo], [Ma]) states that

$$\text{constant term of} \prod_{\substack{1 \le i,j \le n \\ i \ne j}} \left(1 - \frac{x_i}{x_j}\right)^a = \frac{(na)!}{a!^n},$$

where "constant term" means the coefficient of $x_1^0 \ldots x_n^0$ of the Laurent polynomial. The constant term identity of Dyson, as well as its generalizations by Macdonald [Ma] to other root systems (whose proofs were recently completed by Opdam [O]) all involve "evaluating" expressions of the form

$$\text{constant term of } P(x_1, \ldots, x_n)^a,$$

where $P(x_1,\ldots,x_n)$ is a given Laurent polynomial, and more generally,

$$\text{constant term of } P_0(x_1,\ldots,x_n)\prod_{r=1}^{R} P_r(x_1,\ldots,x_n)^{a_r},$$

where P_r are given Laurent polynomials. Replacing the operation of *constant term* by that of *contour integration*:

$$\text{constant term of } f(x_1,\ldots,x_n) = \frac{1}{(2\pi i)^n}\int_C \cdots \int_C \frac{dx_1\cdots dx_n}{x_1\cdots x_n} f(x_1,\ldots,x_n),$$

where C is any circle surrounding the origin, we see that these expressions all have the right format since the integrand

$$F(a_1,\ldots,a_r,x_1,\ldots,x_n) := P_0(x_1,\ldots,x_n)\frac{\prod_{r=1}^{R} P_r(x_1,\ldots,x_n)^{a_r}}{x_1\cdots x_n},$$

is easily seen to be *hypergeometric*. Indeed

$$\frac{F(a_1,a_2,\ldots,a_i+1,\ldots,a_r,x_1,\ldots,x_n)}{F(a_1,a_2,\ldots,a_i,\ldots,a_r,x_1,\ldots,x_n)} = P_i(x_1,\ldots,x_n),$$

which is a rational function, and the logarithmic derivatives of F w.r.t. each of the x_i are also easily seen to be rational functions.

The celebrated Selberg integral [Se] (see also [An1], [Ma])

$$\int_0^1\cdots\int_0^1 \{\prod_{i=1}^{n} t_i^x(1-t_i)^y \prod_{1\le i<j\le n}(t_i-t_j)^{2z}\}dt_1\ldots dt_n = \prod_{j=1}^{n}\frac{(x+(j-1)z)!(y+(j-1)z)!(jz)!}{(x+y+(n+j-2)z+1)!z!}, \quad \text{(Selberg)}$$

and the Mehta-Dyson integral (see [Ma])

$$\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-t_1^2/2-\cdots-t_n^2/2\right)\prod_{1\le i<j\le n}(t_i-t_j)^{2z}dt_1\ldots dt_n = \prod_{j=1}^{n}\frac{(zj)!}{z!}, \quad \text{(Mehta-Dyson)}$$

as well as its various generalizations by Macdonald [Ma] and Richards [Ri] are also obviously of this kind.

Our present work implies that for any *specific* number of variables, there is a short and elegant proof *of any such identity.*

Another application is to combinatorial sums. It is always possible to express a combinatorial sum or multisum as a multi-contour integral of the kind treated here (see [Eg]), and thus in addition to the direct approach to sums, one can go through this roundabout way. The advantage is that we get brand-new *companion identities* quite different from the ones one gets by the direct approach. For example, Dixon's identity (see [Ek2])

$$\sum_k (-1)^k\binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!},$$

can be rewritten as the multi-contour integral

$$\frac{1}{(2\pi i)^3}\int_C dz_1\int_C dz_2\int_C dz_3\frac{(z_1-z_2)^{a+b}(z_3-z_1)^{a+c}(z_2-z_3)^{b+c}}{z_1^{2a+1}z_2^{2b+1}z_3^{2c+1}} = (-1)^{a+b+c}\frac{(a+b+c)!}{a!b!c!}.$$

1.4 Hypergeometric and holonomic functions, and the fundamental lemma.

It's about time that we formally define the notion of *hypergeometric* that we informally introduced above. It will be convenient, at this point, to recall the *shift operators* E_a acting on functions that depend on the discrete variable a (and possibly other variables) by changing a to $a+1$. In symbols:

$$E_a f(a, \mathbf{b}, \mathbf{x}) = f(a+1, \mathbf{b}, \mathbf{x}).$$

Also, as always, D_x will denote the partial derivative w.r.t. x.

Definition. *A function $F(x_1, \ldots, x_n, a_1, \ldots, a_m)$ of n continuous and m discrete variables is a hypergeometric term if for every discrete variable a_i, and every continuous variable x_j,*

$$\frac{E_{a_i} F}{F} = \frac{P_i}{Q_i}, \qquad (i = 1, \ldots, m)$$

and

$$\frac{D_{x_j} F}{F} = \frac{P'_j}{Q'_j} \qquad (j = 1, \ldots, n),$$

where P_i, Q_i, P'_j and Q'_j are all polynomials in the variables $(x_1, \ldots, x_n, a_1, \ldots, a_m)$.

Phrased otherwise, F is a solution of the system of linear recurrence and linear differential equations

$$\begin{aligned} (Q_i E_{a_i} - P_i)F = 0, \qquad & (i = 1, \ldots, m); \\ (Q'_j D_{x_j} - P'_j)F = 0, \qquad & (j = 1, \ldots, n). \end{aligned}$$

Note that these $m+n$ equations are of first order! Functions that satisfy a system of linear differential-recurrence equations with polynomial coefficients, not necessarily of the first order, such that the dimension of the space of solutions of that system is *finite* are called holonomic. The theory of holonomic functions and D-modules (for functions of continuous variables) was initiated by Joseph N. Bernstein [Be], and today is a very active field, see for example [Bo]. It was observed in [Z1] that the theory extends naturally to functions of several discrete and continuous variables, and its importance for proving special function identities is realized. We refer the reader to [Z1] for a leisurely treatment, but the present paper is largely independent of holonomic functions, as will become apparent soon.

For nice things to happen the terms have to be first of all *holonomic*, and for really nice things to happen, they have to be also hypergeometric. Note that being hypergeometric is not enough, and one has to require that our terms be both hypergeometric and holonomic. For example $F(n,k) = 1/(n^2+k^2)$, is hypergeometric but *not* holonomic, as will be proved in the next subsection.

In [Z1] it is shown how to check for holonomicity, and in particular it is proved that the following class of *proper-hypergeometric functions* are holonomic.

Definition. *A term $F(x_1, \ldots, x_n, a_1, \ldots, a_m)$ is proper-hypergeometric if it has the form*

$$P(x_1, \ldots, x_n, a_1, \ldots, a_m) \exp\left\{R_0(x_1, \ldots, x_n)\right\} \prod_{p=1}^{P} S_p(x_1, \ldots, x_n)^{c_p} \times$$
$$\prod_{j=1}^{m} R_j(x_1, \ldots, x_n)^{a_j} \prod_{i=1}^{I} (e_1^{(i)} a_1 + \cdots + e_m^{(i)} a_m + f_i)!^{g_i},$$

where

(i) $P(x_1, \dots, x_n, a_1, \dots, a_m)$ *is a polynomial,*

(ii) R_0, S_p, *and* R_j *are rational functions in* $(x_1, \dots, x_n)$,

(iii) c_p *and* f_i *are commuting indeterminates, or, if one wishes, complex numbers,*

(iv) $e_1^{(i)}, \dots, e_m^{(i)}$ *and* g_i $(i = 1, \dots, I)$, *are (positive or negative) integers.*

Our examples are all proper-hypergeometric. We conjecture that a hypergeometric term is proper-hypergeometric if and only if it is holonomic.

1.5 The fundamental lemma.

Our major tool is the following result, which is due to J. Bernstein, and is reproduced in [Z1], lemma 4.1.

The fundamental lemma. *For every holonomic function* $F(x_1, \dots, x_n, a_1, \dots, a_m)$, *and any continuous variable* x_i *(respectively discrete variable* a_j*), there exist non-zero linear recurrence-differential operators*

$$P_i(x_i; D_{x_1}, \dots, D_{x_n}; E_{a_1}, \dots, E_{a_m}), \quad (\textit{resp. } C_j(a_j; D_{x_1}, \dots, D_{x_n}; E_{a_1}, \dots, E_{a_m}))$$

that annihilate F.

An operator P is said to *annihilate* a function F if $PF = 0$. For example $E_n^2 - E_n - I$ annihilates the Fibonacci sequence, and $D_x^2 + I$ annihilates $\cos x$.

The drawback of the proof in [Z1] is that it doesn't give a priori bounds for the orders (i.e. degrees) in the D's and E's. Hence we also give an *entirely elementary proof* of the fundamental lemma for the special case of *discrete* proper-hypergeometric functions that yields explicit a priori bounds for the orders. We are sure that this proof can be generalized to proper-hypergeometric functions of discrete and *continuous* variables, but the formidable details deter us from carrying it out. At any rate, the a priori bounds are of only theoretical interest, since they are more pessimistic than one obtains in practice.

The main *raison d'être* of the elementary approach is that it generalizes to q-sums/integrals, for which there is no holonomic theory yet, and which is of even greater significance than the ordinary case. Before discussing these, let's fulfill our promise and prove that $F(n, k) = 1/(n^2 + k^2)$ is *not* holonomic, although it's obviously hypergeometric. Were it holonomic, the fundamental lemma would have guaranteed a non-empty set S of non-negative integer pairs (i, j), such that

$$\sum_{(i,j) \in S} \frac{a_{i,j}(n)}{(n+i)^2 + (k+j)^2} = 0,$$

where not all the $a_{i,j}$ are identically zero. Looking at the left side as a meromorphic function of k, we see that at any pole of one term, the other terms remain finite, getting that $\infty =$ finite, a contradiction.

1.6 q-series and integrals.

The simplest non-trivial hypergeometric sequence is $f(n) := n!$, since $f(n)/f(n-1) = n$, and n is the simplest non-constant rational function. As we saw, it is the building block from which all discrete *proper-hypergeometric* functions are built. Combinatorially, $n!$ is the number of permutations of n objects. If instead of naive counting, one assigns to every permutation π, $q^{\mathrm{inv}(\pi)}$, where $\mathrm{inv}(\pi)$, denotes the number of inversions of π, i.e. the number of pairs $1 \le i < j \le n$ such that $\pi(i) > \pi(j)$, then we find (e.g. [An]) that the *weighted count*, denoted by $[n]!$, is

$$[n]! = (1)(1+q)(1+q+q^2)\dots(1+q+q^2+\dots+q^{n-1}),$$

which is called the *q-analog* of $n!$, and it can be rewritten in the form

$$[n]! = \frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)^n}.$$

It turns out that the denominator $(1-q)^n$ is inessential, so one discards it, and instead considers the numerator above, written $(q)_n$, as *the* fundamental *atom* of q-theory. In fact, one needs a slightly more general creature, that emulates $(n+c)!$. It is

$$(c)_n := (1-c)(1-cq)\ldots(1-cq^{n-1}),$$

which agrees with the previous notation when $c = q$. What's nice about it is that $f(n) := (c)_n$, satisfies $f(n+1)/f(n) = (1-cq^n)$, which is the simplest non-constant rational function of q^n. There is also a natural q-analog of the binomial coefficients, probably known to Euler, but called the *Gaussian polynomials*, defined by

$$\binom{n}{m}_q := \frac{(q)_n}{(q)_m(q)_{n-m}} = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-m+1})}{(1-q)\ldots(1-q^m)}. \qquad \text{(Gaussian)}$$

These are much more subtle than the binomial coefficients, since they involve the parameter q, and they have many counting interpretations, both as *generating functions*, in which the coefficients of powers of q count important things, and as functions of $q = p^a$, when they themselves count the m-dimensional subspaces of n-dimensional space over $GF(q)$. We refer the readers to the classic [An], and its sequel [An1], for the combinatorial and analytical aspects of q-theory, as well as for some surprising applications elsewhere in mathematics, and to the soon-to-be-classic [GR], which in addition to the classical q-theory contains a very readable and charming account of the state of the art of q-series, in particular, the impressive work of Askey and Wilson [AW], and the complex yet elegant results of Gasper and Rahman themselves, and that of the many others, who together with Askey, caught the "q-disease".

There are q-analogs, known or conjectured, to almost everything. For example, the q-analog of the binomial theorem, that goes back at least as far as Cauchy. Recall that the the binomial theorem may be written

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k.$$

Its q-analog reads

$$\frac{1}{(x)_{(n+1)}} = \sum_{k=0}^{\infty} \binom{n+k}{n}_q x^k.$$

Other famous examples are the Vandermonde-Chu binomial coefficient identity,

$$\sum_k \binom{a}{k}\binom{b}{k} = \binom{a+b}{a},$$

whose q-analog is

$$\sum_k q^{k^2} \binom{a}{k}_q \binom{b}{k}_q = \binom{a+b}{a}_q,$$

the q-analog of the *Pfaff-Saalschütz* identity

$$\sum_n \binom{a+b+c-n}{a-n,\, b-n,\, c-n,\, n-k,\, n+k} = \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k}, \qquad \text{(Saalschütz)}$$

which is

$$\sum_n q^{(n-k)(n+k)} \binom{a+b+c-n}{a-n, b-n, c-n, n-k, n+k}_q = \binom{a+b}{a+k}_q \binom{a+c}{c+k}_q \binom{b+c}{b+k}_q, \qquad (q\text{-Saalschütz})$$

and the q-analog of Dixon's identity (e.g. [Ek2])

$$\sum_k (-1)^k \binom{a+b}{a+k} \binom{a+c}{c+k} \binom{b+c}{b+k} = \binom{a+b+c}{a, b, c}, \qquad (\text{Dixon})$$

which is

$$\sum_k (-1)^k q^{k(3k+1)/2} \binom{a+b}{a+k}_q \binom{a+c}{c+k}_q \binom{b+c}{b+k}_q = \binom{a+b+c}{a, b, c}_q. \qquad (q\text{-Dixon})$$

Perhaps the most striking q-analog of an ordinary hypergeometric series identity is Watson's q-analog of Whipple's transformation ([Ba], p. 69), that implies the Rogers-Ramanujan identities and many others. Quite recently Gustafson (e.g. [Gu]) and Milne (e.g. [Mi]) separately and together [GuMi], discovered many q-analogs of *multivariate* hypergeometric identities. We will show that all such identities, single-sums, and multi-sums (with a specified number of summations), possess short proofs that can be found by computer. We will give a few examples in the last section. But first we must make the notion of *q-hypergeometric* precise.

So far we have dealt only with discrete functions, but what about continuous ones? Let's look at the function $f(x) := (x)_\infty$. Its claim to fame is that it satisfies the functional equation $f(qx)/f(x) = 1/(1-x)$. It turns out that one can also take $(x)_\infty$ as the "atom" of q-theory, since

$$(q)_k = \frac{(q)_\infty}{(q^{k+1})_\infty}.$$

There are many identities involving products of $(x)_\infty$, or $(x)_k$. These are usually *constant term identities.* We mention here only the q-Dyson identity, conjectured by Andrews and proved in [ZB]. It asserts that (from now on, CT:= "constant term of")

$$CT \prod_{1 \le i < j \le n} (\frac{x_i}{x_j})_{a_i} (\frac{q x_j}{x_i})_{a_j} = \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \ldots (q)_{a_n}}. \qquad (q\text{-Dyson})$$

We will show that (q-Dyson) and any other identity of the same genre, for a *fixed* number of variables, possesses a computer-generated one-line proof.

Recall that E_{k_i} denotes the shift operator in the k_i variable, obtained by incrementing k_i by 1. Motivated by the fact that $(qx)_\infty/(x)_\infty$ is nice (i.e. $1/(1-x)$), we also introduce the *dilation* operators for every variable x by

$$Q_x f(x, \mathbf{y}) := f(qx, \mathbf{y}).$$

From these one can form the q-derivative

$$D_x^{(q)} f(x, \mathbf{y}) = \frac{f(qx, \mathbf{y}) - f(x, \mathbf{y})}{(q-1)x} = \frac{(Q_x - 1) f(x, \mathbf{y})}{(x(q-1))},$$

which by L'Hospital's and the chain rules, tends to D_x, as $q \to 1$.

It is now clear what the definition of *q-hypergeometric* should be.

Definition. *A term $F(k_1,\dots,k_r,y_1,\dots,y_s)$ in r discrete variables $\mathbf{k}$ and s continuous variables $\mathbf{y}$, is q-hypergeometric if for every discrete variable k_i, $(E_{k_i}F)/F$ and for every continuous variable y_j, $(Q_{y_j}F/F)$, all are rational functions of $(q^{k_1},\dots,q^{k_r},y_1,\dots,y_s)$ and possibly other constant parameters, including q.*

As of this writing, we do not know of any q-extension of the holonomic theory, although it appears that it should be possible, especially in view of Antony Joseph's (see [Eh], p. 178) beautiful and short proof of Bernstein's inequality that seems to be q-generalizable. Be that as it may, we will stick to the elementary approach, and henceforth consider only *q-proper-hypergeometric* terms.

It is clear what should be included. Polynomials

$$P(q^{k_1},\dots,q^{k_r},y_1,\dots,y_s) \tag{qPH-I}$$

should obviously qualify, as should any expression of the form

$$(cy_1^{\alpha_1}\dots y_s^{\alpha_s}q^{\beta_1k_1}\dots q^{\beta_rk_r})_\infty^\gamma, \tag{qPH-II}$$

where the α_i and β_j and γ are (positive or negative) *integers*, and c is any commuting indeterminate constant or parameter. Then there is one more creature that is not the q-analog of anything ordinary, or if one wishes, is the q-analog of 1. It is q raised to any quadratic polynomial, with integer coefficients, in $k_1,\dots,k_r$, i.e.,

$$q^{\sum_{i,j}a_{i,j}k_ik_j+\sum_i b_ik_i}, \tag{qPH-III}$$

where the $a_{i,j}$ and the b_i are (positive or negative) integers or half-integers. Finally there are expressions

$$z_1^{k_1}\dots z_r^{k_r} \tag{qPH-IV}$$

We are now ready to define *q-proper-hypergeometric terms.*

Definition. *A term $F(k_1,\dots,k_r,y_1,\dots,y_s)$ that involves r discrete variables and s continuous variables is q-proper-hypergeometric if it is a product of an expression of type (qPH-I) (i.e. a polynomial), of an expression of type (qPH-III) (i.e. q raised to a quadratic form), of a monomial (qPH-IV), and of any finite number of expressions of type (qPH-II).*

In section 5 we will prove the following fundamental result.

The q-fundamental lemma. *For every q-proper-hypergeometric term*

$$F(x_1,\dots,x_n,a_1,\dots,a_m),$$

and every continuous variable x_i (respectively discrete variable a_j), there exist non-zero linear recurrence-q-differential operators

$$P_i(x_i;Q_{x_1},\dots,Q_{x_n};E_{a_1},\dots,E_{a_m})(\text{ resp. } C_j(q^{a_j};Q_{x_1},\dots,Q_{x_n};E_{a_1},\dots,E_{a_m})),$$

annihilating F.

Since a continuous variable x can be converted to a discrete variable, by setting $x=q^k$ and using $(cq^k)_\infty=(c)_\infty/(c)_k$, we can w.l.o.g. assume that $n=0$, i.e., consider only functions of discrete variables. Of course, we could have taken on the other extreme, but given our predilections, we prefer to be discrete, whenever possible.

After the discrete version of the operator is found, one gets back the original operator by substituting q^k for x and E_k for Q_x. This is legitimate because our proof and our algorithm use only the commutation relation $E_kq^k=qq^kE_k$, which is "isomorphic" to $Q_xx=qxQ_x$. In the accompanying certificate (see below) we substitute $(c)_k$ for $(c)_\infty/(cx)_\infty$.

2. The fundamental theorem of hypergeometric summation/integration

2.1 Introduction.

We need two more concepts.

Definition. *A function $F(k_1,\dots,k_r,y_1,\dots,y_s)$ vanishes at infinity, if for every variable k_i and y_j,*

$$\lim_{|k_i|\to\infty} F(\mathbf{k},\mathbf{y})=0,\ \lim_{|y_j|\to\infty} F(\mathbf{k},\mathbf{y})=0.$$

Definition. *An integral-sum*

$$g(\mathbf{n},\mathbf{x}):=\sum_{\mathbf{k}}\int_{\mathbf{y}} F(\mathbf{n},\mathbf{k},\mathbf{x},\mathbf{y})d\mathbf{y}, \qquad (\textit{general-integral-sum})$$

is pointwise trivially evaluable, if for every specific specialization of the auxiliary variables (parameters) $\mathbf{n},\mathbf{x}$ there is an algorithm that will evaluate it.

For example, in the case of pure sums that are *terminating*

$$g(\mathbf{n}):=\sum_{\mathbf{k}} F(\mathbf{n},\mathbf{k}), \qquad (\text{general-sum})$$

where the sum is always finite for every specific choice of $\mathbf{n}$ (since $F(\mathbf{n},\cdot)$ has finite support), we have both properties.

We are now ready for the *fundamental theorem of hypergeometric summation-integration.*

The fundamental theorem. *Let Δ_{k_i} denote the forward difference operator in k_i: $\Delta_{k_i}:=(K_i-1)$. Let $F(n,k_1,\dots,k_r,y_1,\dots,y_s)$ $(resp. F(x,k_1,\dots,k_r,y_1,\dots,y_s))$ be hypergeometric and holonomic (both hold if it is proper-hypergeometric) in $(\mathbf{k},\mathbf{y})$ and n (resp. x), where $n,\mathbf{k}$ are discrete variables and $x,\mathbf{y}$ are continuous variables. There exists a linear ordinary recurrence (resp. differential) operator with polynomial coefficients $P(N,n)$ (resp. $P(D_x,x)$), and rational functions (in all the variables) $R_1,\dots,R_r,S_1,\dots,S_s$ such that*

$$P(N,n)F \quad (\textit{resp. } P(D_x,x)F)=\sum_{i=1}^{r}\Delta_{k_i}(R_iF)+\sum_{j=1}^{s}D_{y_j}(S_jF). \qquad (2.1.1)$$

Furthermore, if F is *proper-hypergeometric*, it is possible to find a priori bounds for the order of $P(N,n)$ (resp. $P(D_x,x)$), and it is possible to predict denominators for R_i and S_j.

It is easy to see that once the operator $P(N,n)$ (or $P(D_x,x)$) and the rational functions R_i and S_i are given, the verification of (2.1.1) is a purely routine matter. Indeed, dividing through by F, and taking advantage of the hypergeometric form, it is seen that (2.1.1) reduces, in any given instance, to a finite identity involving sums and products of rational functions, and hence, by clearing denominators, of polynomials. Of course, thanks to our algorithm, the generation of (2.1.1), in *any given case*, is also a purely routine matter.

Proof of the fundamental theorem. By the fundamental lemma, there exists an operator

$$A(n;E_n,E_{k_1},\dots,E_{k_r};D_{y_1},\dots,D_{y_s})$$

annihilating F. It is possible to write (in many ways):

$$A(n; E_n, E_{k_1}, \dots, E_{k_r}; D_{y_1}, \dots, D_{y_s}) = P(n, E_n) - \sum_{i=1}^{r} (E_{k_i} - I) B_i(n; E_n, E_{k_1}, \dots, E_{k_r}; D_{y_1}, \dots, D_{y_s})$$
$$- \sum_{j=1}^{s} D_{y_j} \bar{B}_j(n; E_n, E_{k_1}, \dots, E_{k_r}; D_{y_1}, \dots, D_{y_s}). \tag{2.1.2}$$

Now apply F to the above operator equation. Since the operator A annihilates F, we get that

$$\begin{aligned} 0 = P(n, E_n)F &- \sum_{i=1}^{r} (E_{k_i} - I) B_i(n, E_n, E_{k_1}, \dots, E_{k_r}; D_{y_1}, \dots, D_{y_s})F \\ &- \sum_{j=1}^{s} D_{y_j} \bar{B}_j(n, E_n, E_{k_1}, \dots, E_{k_r}; D_{y_1}, \dots, D_{y_s})F. \end{aligned} \tag{2.1.3}$$

Since F is hypergeometric, $E_{k_i}F/F$, and $D_{y_j}F/F$ are rational functions. It follows by induction that for any "operator monomial"

$$Mon := E_n^{\alpha_0} \prod_{i=1}^{r} E_{k_i}{}^{\alpha_i} \prod_{j=1}^{s} D_{y_j}{}^{\beta_j} \tag{Op-Mon}$$

$(MonF)/F$ is a rational function. Hence for *any* operator

$$T(n, k_1, \dots, k_r; y_1, \dots, y_s; E_n, E_{k_1}, \dots, E_{k_r}; D_{y_1}, \dots, D_{y_s}),$$

TF/F is a rational function, since T is a linear combination with coefficients that are polynomials in all of the variables, of operator monomials. In particular,

$$B_i(F) = R_i F, \quad \bar{B}_j(F) = S_j F,$$

for some *rational functions* R_i and S_j. Putting these in (2.1.3) completes the proof. The case in which the auxiliary variable is x rather than n is similar. $\square$

In proving the fundamental theorem, we used the fact that we could find an operator annihilating F of the form

$$P(N, n) - \sum_{i=1}^{s} \Delta_i B_i - \sum_{j=1}^{r} D_{y_j} \bar{B}_j.$$

If $P(N, n)$ (or $P(D_x, x)$) happens to be the zero operator, then it is possible to use (2.1.1) to get another, nonzero, operator that annihilates $f(n)$ (resp. $f(x)$.)

Because of the fundamental lemma, we had that the B_i and $\bar{B}_j$ had the additional "nice" property that they were independent of $(k_1, \dots, k_r; y_1, \dots, y_s)$. Gert Almkvist observed that this luxury is unnecessary for the fundamental theorem to work! Although both the proofs in [Z1] and the new elementary proof for proper-hypergeometric functions given in section 4 are constructive, they yield far from optimal operators $P(N, n)$ and accompanying certificates R_i, S_j, and hence are mainly of theoretical value.

The significance of the fundamental theorem is manifest from its

Fundamental corollary. *If $F(n,\mathbf{k},\mathbf{y})$ (resp. $F(x,\mathbf{k},\mathbf{y})$) is as above, and vanishes at infinity for every fixed n (resp. x), then*

$$f(n) := \sum_{\mathbf{k}} \int F(n,\mathbf{k},\mathbf{y})d\mathbf{y} \quad (resp.\ f(x) := \sum_{\mathbf{k}} \int F(x,\mathbf{k},\mathbf{y})d\mathbf{y}) \tag{2.1.4}$$

satisfies a linear recurrence (resp. differential) equation with polynomial coefficients:

$$P(N,n)f(n) = 0 \quad (resp.\ P(D_x,x)f(x) = 0). \tag{2.1.5}$$

Proof. Sum-integrate (2.1.1) w.r.t. $k_1,\ldots,k_r,y_1,\ldots,y_s$. □

Extending the terminology of [WZ2], we call the rational functions R_i, S_j the *certificates* of the identity (2.1.5).

2.2 How to find $P(N,n)$ and the certificates.

Now that we have the theoretical certitude that there *exist* a $P(N,n)$ and certificates R_i, S_j that are rational functions, such that (2.1.1) is true, we can use (2.1.1) itself to find them! All we do is use the method of undetermined coefficients. We first "guess" the order of $P(N,n)$, say L, and set

$$P(N,n) = \sum_{i=0}^{L} b_i(n)N^i, \tag{2.2.1}$$

where $b_i(n)$ are as yet unknown polynomials (or rational functions) in n.

In practice there is no guessing, we just start with the optimistic extreme $L = 0$, try it, and work our way up. We know that eventually we will be successful, and, for discrete proper-hypergeometric functions, we can find an effective upper bound for L (see theorem 4.1 below). Next we have to "guess", or rather predict, the denominators of the rational functions R_i and S_j. This is always possible, since by looking at the functional equation obtained from (2.1.1), by plugging in (2.2.1) and then dividing throughout by F, it is possible to predict both the "worst" and the "best" conceivable denominators for the certificates R_i, S_j. Then one tries out all the possibilities from best to worst. Details, describing various shortcuts, will appear in the above-mentioned forthcoming paper. Finally, we determine the "generic" degrees of the polynomials that are the numerators of the R_i and S_j, express them generically, with undetermined coefficients, and put it all in the above-mentioned functional equation implied by (2.1.1). We then clear denominators, and equate coefficients of all the monomials

$$k_1^{\alpha_1}\ldots k_r^{\alpha_r}y_1^{\beta_1}\ldots y_s^{\beta_s}.$$

What results is a huge system of linear equations in the unknowns b_i and the coefficients of the numerators of the certificates, over the "ground field" of rational functions in n. If we find a nonzero solution, we are done. If not, just make the denominators "bigger", or increase L. The fundamental theorem promises you eventual success.

Luckily, in most real life examples the computer time is not prohibitive, as we will illustrate with the numerous examples given in section 6.

2.3 How to prove identities fast: WZ-tuples.

Suppose we have to prove an identity of the form

$$\sum_{\mathbf{k}} \int F(n, \mathbf{k}, \mathbf{y}) d\mathbf{y} = \sum_{\mathbf{k}'} \int G(n, \mathbf{k}', \mathbf{y}') d\mathbf{y} \qquad \text{(general-identity)}$$

Let's call the left side $\text{Left}^{(n)}$ and the right side $\text{Right}^{(n)}$. We find an operator $P(N,n)$ annihilating $\text{Left}^{(n)}$ and an operator $\bar{P}(N,n)$ annihilating $\text{Right}^{(n)}$, with the appropriate certificates. If, as is usually the case, P and $\bar{P}$ are identical, this proves the identity once the initial conditions $\text{Left}^{(n)} = \text{Right}^{(n)}$, $n = 0, 1, \ldots, L-1$ are checked. This is always trivial if, as we assume, the integral-sums in question are pointwise trivially evaluable. In the rare event that P and $\bar{P}$ are different, one can use the Euclidean algorithm (adapted to the non-commutative ring of linear recurrence operators with polynomial coefficients) to find a "minimal" operator $A(N,n)$ such that both P and $\bar{P}$ are left multiples of it. It follows that both $\text{Left}^{(n)}$ and $\text{Right}^{(n)}$ are annihilated by $A(N,n)$ if it is true up to $n = \max(order(P), order(\bar{P}))$. In the above, we tacitly assumed that the coefficients of the leading terms of both P and $\bar{P}$ do not have positive integer zeroes. It they do, just shift the starting value of n to the largest such, and check the identity case by case until then (we do not know of any case where this actually happens).

The treatment for integral-sums whose free variable is a continuous x rather than a discrete n is similar. If, as in the "general integral-sum" the free variables are multi: $\text{Left}(\mathbf{n}, \mathbf{x})$, $\text{Right}(\mathbf{n}, \mathbf{x})$, then one does things for every single variable separately, or builds up one variable at a time.

A very important special case is that in which $\text{Right}^{(n)}$ is a plain hypergeometric term. In that case we can divide by it, and get an identity of the form

$$\sum_{\mathbf{k}} \int F(n, \mathbf{k}, \mathbf{y}) d\mathbf{y} = 1 \qquad \text{(explicitly-evaluable)}$$

Now we need to work only with the left side. Getting the operator $P(N,n)$ and checking that indeed $\text{Left}^{(n)} = 1$ for $n = 0, \ldots, order(P)-1$, all we have to prove is that $P(N,n)$ annihilates the constant sequence 1. In other words, writing

$$P(N,n) = \sum_{i=0}^{L} b_i(n) N^i,$$

we have to check that

$$\sum_{i=0}^{L} b_i(n) = 0.$$

But, if all goes well, $P(N,n)$ would be of minimal order, i.e. a multiple by a scalar of the *minimal* operator annihilating 1, which is $I - N$, so $L = 1$, and $P(N,n) = b_0(n)(I - N)$. Absorbing the factor $b_0(n)$ inside the certificates, (2.1.1) becomes

$$\Delta_n F + \sum_{i=1}^{r} \Delta_{k_i}(R_i F) + \sum_{j=1}^{s} D_{y_j}(S_j F) = 0. \qquad \text{(WZ-tuple)}$$

We call the tuple $(F, R_1F, \ldots, R_rF; S_1F, \ldots, S_sF)$ a *WZ-tuple*. This is a natural generalization of the notion of *WZ pair* introduced in [WZ2]. Recall that any WZ pair produced *two* (terminating) identities, the original one, and a new one, which we called the *companion*. A WZ-tuple yields $1 + r + s$ identities: the original one, and $r + s$ "bonuses". The proofs are all obtained in the same stroke. Once we know that we have a WZ-tuple, summing-integrating w.r.t. all the variables save the preferred one, yields it. So the $r+s$ bonus identities are

$$\sum_{k_1,\dots,\hat{k}_i,\dots k_r} \int (R_i F) d\mathbf{y} = \text{Constant} \qquad (i=1,\dots,r) \qquad \text{(bonus-discrete)}$$

and

$$\sum_{\mathbf{k}} \int (S_j F) dy_1 dy_2 \dots \widehat{dy_j} \dots dy_s = \text{Constant}, \qquad (j=1,\dots,s). \qquad \text{(bonus-continuous)}$$

The only catch is that now the integral-sum might diverge, i.e. the "Constant" equals ∞. This is easily overcome by the process of shadowing described in [WZ2], that finds an "equivalent" WZ-tuple for which the integral-sum of interest is pointwise trivially evaluable.

The theme of WZ-tuples is explored extensively in [Z4], which, however lacked the flesh and blood of examples. The present algorithm provides them amply.

2.4 The q-case.

The statement, and proof, of the fundamental theorem for q-proper-hypergeometric functions follow almost verbatim. Also all the discussion above applies. Only now we must restrict to *proper q-hypergeometric* functions.

The q-fundamental theorem. *Let Δ_{k_i} denote the forward difference operator in k_i: $\Delta_{k_i} := (K_i - 1)$; for any variable y, let Q_y be the q-dilation operator in that variable: $Q_y f(y) := f(qy)$. Let $F(n,\mathbf{k},\mathbf{y})$ (resp. $F(x,\mathbf{k},\mathbf{y})$) be a q-proper-hypergeometric function in $(\mathbf{k},\mathbf{y})$ and n (resp. x), where $n,\mathbf{k}$ are discrete variables and $x,\mathbf{y}$ are continuous variables. Then there exists a linear ordinary recurrence (resp. q-differential) operator with polynomial coefficients (the polynomials being in (q^n,q) or (x,q) respectively) $P(N,q^n,q)$ (resp. $P(Q_x,x,q)$), and rational functions (in $(q^n,q^{k_1},\dots,q^{k_r},y_1,\dots,y_s,q)$) $R_1,\dots,R_r,S_1,\dots,S_s$ such that*

$$P(N,q^n,q)F \quad (\textit{resp. } P(Q_x,x,q)F) = \sum_{i=1}^{r} \Delta_{k_i}(R_i F) + \sum_{j=1}^{s} (Q_{y_j} - I)(S_j F). \tag{2.4.1}$$

Furthermore, it is possible to find a priori bounds for the order of $P(N,q^n,n)$ (resp. $P(Q_x,x,q)$), and the denominators of R_i and S_j.

The q-fundamental theorem follows from the *q-fundamental lemma* stated at the end of section 1.6, the same way that the fundamental Theorem followed from the fundamental Lemma. Like its ordinary counterpart, it implies the following corollary, in which $CT_{\mathbf{y}} f(\mathbf{y})$ denote the coefficient of $y_1^0 \dots y_n^0$ in the Laurent polynomial $f(\mathbf{y})$. However the result still holds if $f(\mathbf{y})$ is something else and $CT_{\mathbf{y}}$ is replaced by $\int (f(y)/y) dy$ provided the integral makes sense. Of course $CT_{\mathbf{y}}$ is the case where the integral is a contour integral around the origin, aside from a constant factor.

q-fundamental corollary. *If $F(n,\mathbf{k},\mathbf{y})$ (resp. $F(x,\mathbf{k},\mathbf{y})$) is q-proper-hypergeometric, and vanishes at infinity for every fixed n (resp. x), then*

$$f(n) := \sum_{\mathbf{k}} CT_{\mathbf{y}} F(n,\mathbf{k},\mathbf{y}) \quad (\textit{resp.} f(x) = \sum_{\mathbf{k}} CT_{\mathbf{y}} F(x,\mathbf{k},\mathbf{y})) \tag{2.4.2}$$

satisfies a linear recurrence (resp. q-differential) equation with polynomial coefficients in (q^n,q) (resp. (x,q)):

$$P(N,q^n)f(n) \equiv 0 \quad (\textit{resp. } P(Q_x,x)f(x) \equiv 0). \tag{2.4.3}$$

The discussion following the fundamental corollary, about finding the $P(N,q^n)$ and the certificates, applies almost verbatim to this case, and is left to the readers.

3. Recurrence operators via Sister Celine's technique

In this section and the next we will pursue the elementary and explicit approach to single and multivariate hypergeometric summation, specifically to the proof of the fundamental lemma. As we already pointed out, this yields explicit a priori bounds for the order and "size" of the certificates, and extends naturally (section 5) to the q-case, for which it gives the only known proof (for proper q-hypergeometric summation and integration) of the q-fundamental lemma.

The present approach is a systematization, generalization, and quantification of the fundamental work of Sister Mary Celine Fasenmyer [Fa], and also builds on work of Verbaeten [V]. We repeat the *caveat* issued in [Z1], that parts of [Z0] are erroneous. The present paper corrects those errors.

Another elementary approach to proving that single and multivariate proper hypergeometric sums satisfy linear recurrence equations with polynomial coefficients can be pursued by Lipshitz's [L] powerful theorem. This approach, while it is explicit in principle, in fact yields an infeasible algorithm, and doesn't prove the fundamental theorem with certificates; only the fundamental corollary. Furthermore it does not seem to extend to q-sums.

3.1 The k-free recurrence in the case of one variable.

We begin with the case of a single variable of summation. Although it is known that such sums satisfy recurrence relations, a study of this case will serve to introduce the methods of sections 3, 4, and 5 of this paper. These methods will be essentially the same in the cases of multivariate, q, etc. identities, though their implementation will become more demanding there. Even in the one-variable case, however, we will obtain some new results, namely explicit bounds for the orders of the recurrence relations whose existence is guaranteed by the theory.

Definition. *A proper-hypergeometric term is a function of the form*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{pp}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{qq}(u_s n + v_s k + w_s)!}\xi^k, \tag{3.1.1}$$

where P is a polynomial and ξ is a parameter. The a's, b's, u's and v's are assumed to be specific integers, i.e., they are integers and do not depend on any other parameters. The c's and the w's are also integers, but they may depend on parameters. We will say that F is well-defined at (n,k) if none of the numbers $\{a_s n + b_s k + c_s\}_1^{pp}$ is a negative integer. We will say that $F(n,k) = 0$ if F is well-defined at (n,k) and at least one of the numbers $\{u_s n + v_s k + w_s\}_1^{qq}$ is a negative integer, or $P(n,k) = 0$.

The word 'proper' in the above definition is intended to underscore the absence of a denominator polynomial in (3.1.1). If additional parameters are present in the c's and/or the w's then the conditions that F be well-defined, be nonzero, etc. will translate into certain restrictions on the allowable values of those parameters.

Though our goal is to obtain recurrence relations for definite *sums*, our starting point will always be to obtain a certain kind of recurrence relation (*k-free recurrence*) for the proper-hypergeometric term F itself. This approach has a number of advantages. First, the conditions on F that insure that it possesses a k-free recurrence are very mild. Second, with some extra conditions we will be able to deduce recurrences that are satisfied by polynomials whose coefficients are the given term, and for sums of values of the term. Third, the k-free recurrence for F will be an excellent starting point for the telescoping certification of identities (sec. 3.2), and will streamline the derivations there. Fourth, the difficulties of multivariable and q generalizations will be minimized.

Definition. *A proper-hypergeometric term F is said to satisfy a k-free recurrence at a point $(n_0, k_0) \in \mathbb{Z}^2$ if there are integers I, J and polynomials $\alpha_{i,j} = \alpha_{i,j}(n)$ that do not depend on k and are not all zero, such that the relation*

$$\sum_{i=0}^{I}\sum_{j=0}^{J} \alpha_{i,j}(n) F(n-j, k-i) = 0 \tag{3.1.2}$$

holds for all (n, k) in some $\mathbb{R}^2$ neighborhood of (n_0, k_0), in the sense that F is well-defined at all of the arguments that occur, and the relation (3.1.2) is true.

The main result of the present subject is the following.

Theorem 3.1. *Every proper-hypergeometric term F satisfies a nontrivial k-free recurrence relation. Indeed there exist I, J and polynomials $\alpha_{i,j}(n)$ $(i = 0, \ldots, I;\ j = 0, \ldots, J)$ not all zero, such that (3.1.2) holds at every point $(n_0, k_0) \in \mathbb{Z}^2$ for which $F(n_0, k_0) \neq 0$ and all of the values $F(n_0 - j, k_0 - i)$ that occur in (3.1.2) are well-defined. Furthermore there exists such a recurrence with $(I, J) = (I^*, J^*)$, where*

$$J^* = \sum_s |b_s| + \sum_s |v_s|, \qquad I^* = 1 + deg(P) + J^*(\{\sum_s |a_s| + \sum_s |u_s|\} - 1). \tag{3.1.3}$$

The best known example of a k-free recurrence for a proper-hypergeometric term is undoubtedly the Pascal triangle recurrence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

which holds at every grid point in the plane other than the origin. As a slightly less trivial example we will use the recurrence

$$n\binom{n}{k}^2 - (2n-1)\left\{\binom{n-1}{k}^2 + \binom{n-1}{k-1}^2\right\} + (n-1)\left\{\binom{n-2}{k}^2 - 2\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}^2\right\} = 0, \tag{3.1.4}$$

for the squares of the binomial coefficients, which can be recognized as a disguised form of the familiar recurrence for the Legendre polynomials.

We now prove theorem 3.1. Fix some $I, J > 0$, and suppose (n_0, k_0) is a point that satisfies the two conditions of the theorem. Since we assumed that all of the a_s, b_s, u_s, v_s in (3.1.1) are integers, we have that for all (n, k) in some $\mathbb{R}^2$ neighborhood of (n_0, k_0), all of the ratios $F(n-j, k-i)/F(n, k)$ are well-defined *rational functions* of n and k. Hence we can form a linear combination

$$\sum_{i=0}^{I}\sum_{j=0}^{J} \alpha_{i,j}(n)\frac{F(n-j, k-i)}{F(n, k)} \tag{3.1.5}$$

of these rational functions, in which the α's are to be determined, if possible, so as to make the sum vanish identically in the neighborhood.

The idea is the following. We will find a common denominator for the summands in (3.1.5), and will express (3.1.5) as a single polynomial in k whose coefficients involve n (polynomially) and all of the α's (linearly), divided by that common denominator. To make (3.1.5) vanish identically it will suffice to make the coefficient of every power of k that occurs in the numerator polynomial vanish, by choosing the α's appropriately. This will yield a system of linear homogeneous equations to solve for the α's, and for which we will want to assert that a nontrivial solution exists. To do that we will compare the number of equations that must be satisfied, which is 1 greater than the degree in k of the numerator polynomial, with the number of α's that are available, namely $(I+1)(J+1)$, and show that I and J can be chosen so that the latter exceeds the former.

Next we give the details about what all of these polynomials look like. To do that we define the functions rf, prf, ff, pff, in which the names are to be suggestive of 'rising factorial', 'partial rising factorial', etc.:

$$\mathrm{rf}(x,y) := \prod_{j=1}^{x}(y+j) \tag{3.1.6)(a}$$

$$\mathrm{prf}(x,y,u) := \prod_{j=x+1}^{y}(u+j) \tag{3.1.6)(b}$$

$$\mathrm{ff}(x,u) := \prod_{j=0}^{x-1}(u-j) \tag{3.1.6)(c}$$

$$\mathrm{pff}(x,y,u) := \prod_{j=x}^{y-1}(u-j) \tag{3.1.6)(d}$$

$$p_1(i,j,k) := \prod_{r=1}^{pp}\left\{\mathrm{rf}((-ja_r-ib_r)^+, a_rn+b_rk+c_r)\mathrm{pff}((ja_r+ib_r)^+, J(a_r)^+ + I(b_r)^+, a_rn+b_rk+c_r)\right\} \tag{3.1.6)(e}$$

$$p_2(i,j,k) := \prod_{s=1}^{qq}\left\{\mathrm{ff}((ju_s+iv_s)^+, u_sn+v_sk+w_s)\mathrm{prf}((-ju_s-iv_s)^+, J(-u_s)^+ + I(-v_s)^+, u_sn+v_sk+w_s)\right\} \tag{3.1.6)(f}$$

For a real number x we define $x^+ = \max\{0, x\}$.

In terms of these functions the numerator polynomial of the sum in (3.1.5), after finding a common denominator and collecting all terms over that denominator, is

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{ij}(n)P(n-j,k-i)p_1(i,j,k)p_2(i,j,k). \tag{3.1.7}$$

It should be emphasized that every term in (3.1.7) is a polynomial in k, all cancellations and divisions having been done in advance. The degree of (3.1.7), as a polynomial in k is therefore at most

$$I\left\{\sum_r |b_r| + \sum_s |v_s|\right\} + J\left\{\sum_r |a_r| + \sum_s |u_s|\right\} + \deg(P),$$

$= \gamma I + \delta J + \epsilon$, say.

The number of parameters α_{ij} that are available is $(I+1)(J+1)$, and the number of linear, homogeneous conditions that they must satisfy is $1 + \gamma I + \delta J + \epsilon$.[3] Hence a nontrivial set of α's exists if $(I+1)(J+1) > 1 + \gamma I + \delta J + \epsilon$. This inequality surely holds for all sufficiently large I provided that $J \geq \gamma$. Further, if $J = \gamma$, then the inequality holds provided $I \geq \gamma(\delta - 1) + 1 + \epsilon$. Hence there surely exists a nontrivial recurrence of orders

$$J^* = \sum_r |b_r| + \sum_s |v_s|, \qquad I^* = 1 + \deg(P) + J^*\left(\left\{\sum_r |a_r| + \sum_s |u_s|\right\} - 1\right),$$

and the proof of theorem 3.1 is complete.

[3] It is precisely here that we use the lack of a denominator polynomial in the assumed form (3.1.1) of F. Certain kinds of denominator polynomials would be admissible, and the question of characterizing them is an interesting one.

3.2 The certification of identities in one summation variable.

In the previous section we established that proper-hypergeometric terms themselves satisfy k-free linear recurrences with polynomial-in-n coefficients. Now we will give one of the corollaries of that fact, which is a very effective *certification procedure* for identities.

To certify an identity is to give some additional information, beyond the identity itself, that will enable another person to verify that the identity is true much more easily than if the additional information (*certificate*) were not available. The second person has the tasks of first showing that the alleged certificate is true, and then showing that the certificate implies the identity in question.

The main result on the certification of identities in one variable of summation is the following.

Theorem 3.2A. *Let F be a proper-hypergeometric term, and let $(n, k) \in \mathbb{Z}^2$ be a point at which $F(n, k) \neq 0$ and such that $F(n-j, k-i)$ is well-defined for all $0 \leq i \leq I$ and $0 \leq j \leq J$. Then there are polynomials $a_0(n), \ldots, a_J(n)$, not all zero, and a function $G(n, k)$ such that $G(n, k) = R(n, k)F(n, k)$ for some rational function R and such that*

$$a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = G(n,k) - G(n,k-1). \tag{3.2.1}$$

Proof. Let there be given a k-free recurrence for some proper-hypergeometric term F. We write the recurrence, using operator notation, in the form $H(N, K, n)F(n, k) = 0$, where the backwards shift operators N and K are defined by $Ng(n, k) = g(n-1, k)$ and $Kg(n, k) = g(n, k-1)$. Since H is a polynomial in its arguments we can expand it as

$$H(N, K, n) = H(N, 1, n) + (K-1)V(N, K, n).$$

Thus we have

$$\begin{aligned} 0 &= H(N,K,n)F(n,k) = H(N,1,n)F(n,k) + (K-1)V(N,K,n)F(n,k) \\ &= H(N,1,n)F(n,k) + (K-1)G(n,k) \qquad \text{(say)} \\ &= H(N,1,n)F(n,k) + G(n,k-1) - G(n,k). \end{aligned} \tag{3.2.2}$$

Thus we have $H(N, 1, n)F(n, k) = G(n, k) - G(n, k-1)$, which is of the form (3.2.1). Furthermore G is a rational multiple of F because VF is a linear combination, with polynomial coefficients, of the values of F that occur in its k-free recurrence. Because of the form (3.1.1) of F, each of these $F(n-j, k-i)$ is a rational multiple of $F(n, k)$, and the proof is complete. □

Example A. In the case $F(n,k) = \binom{n}{k}^2$ the k-free recurrence is (3.1.4). We rewrite it using the operators N and K as

$$\{n - (2n-1)(N + KN) + (n-1)(N^2 - 2N^2 + N^2K^2)\}F(n,k) = 0,$$

which defines the operator $H(N, K, n)$ of (3.2.2). We can expand H as

$$\begin{aligned} H(N,K,n) &= H(N,1,n) + (K-1)V(N,K,n) \\ &= \{n - 2(2n-1)N\} + (K-1)\{-(2n-1)N + (n-1)N^2(K-1)\}. \end{aligned}$$

Thus the assertion (3.2.1) of theorem 3.2A becomes, in this example,

$$n\binom{n}{k}^2 - 2(2n-1)\binom{n-1}{k}^2 = G(n,k) - G(n,k-1) \tag{3.2.3}$$

where

$$G(n,k) = V(N,K,n)F(n,k) = \{-(2n-1)N + (n-1)N^2(K-1)\}\binom{n}{k}^2$$
$$= \{(2+2k-3n)(1-\frac{k}{n})^2\}\binom{n}{k}^2.$$

We can already see that (3.2.3) is a certification for the familiar identity

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}. \tag{3.2.4}$$

For first of all it is easy to check that (3.2.3) is true, by dividing through by $\binom{n}{k}^2$ and verifying the resulting rational function identity. Second it is easy to prove (3.2.4) from the certificate (3.2.3): just sum (3.2.3) over, say, $k = 0, \ldots, n$, and notice the telescoping on the right side. The result will be a recurrence *of order 1* for the sum. Finally, just check that the right side of (3.2.4) satisfies the same recurrence and the proof will be complete. In this example, of course, the amount of work was more than the identity "deserved", but the method is completely general, and works every time. $\square$

3.3 Standard boundary conditions.

So far the theorems have been about the summand itself. The main objects of study, of course, are definite sums over k of $F(n,k)$, and we now turn our attention to them. This subject subdivides according to whether the limits of the sum include all nonzero values of the summand or not. Thus in the sum $\sum_{k=0}^{n}\binom{n}{k}$ all nonzero values are included, so we can rewrite such a sum as $\sum_k \binom{n}{k}$ without confusion. In such cases we speak of *standard boundary conditions*, in which the phrase refers to the fact that the summand vanishes outside the range of summation.

In other cases, such as in $\sum_{k=0}^{n}\binom{3n}{k}$, the limits of the sum bear no discernible relationship to the vanishing or nonvanishing of the summand, and we speak of *nonstandard boundary conditions.*

In both cases the attack on the problem begins with finding the k-free recurrence for the summand, as in the previous section. In the case of standard boundary conditions, summation of that recurrence will yield a homogeneous recurrence relation with polynomial-in-n coefficients, for the unknown sums. In the case of nonstandard boundary conditions we will obtain an inhomogeneous recurrence for the sums.

In this section we discuss the case of standard boundary conditions and the associated sums. The nicest way to get at the sums, it turns out, is through an intermediate step that is not without intrinsic interest: hypergeometric series. Indeed, finding recurrences for such series was the main motivation of Sister Celine [Fa] in developing some of these techniques. If F is a proper-hypergeometric term, then associated with F there is the hypergeometric series $\sum_k F(n,k)x^k$, in which we will now discuss the limits of the summation.

For a fixed integer n, we let $B(n) = [a(n), b(n)]$ denote a maximal interval of integer values of k for which $F(n,k)$ is well-defined and nonzero. Just outside of the interval $B(n)$ we suppose that there are intervals $\alpha(n) \le k < a(n)$ and $b(n) < k \le \beta(n)$ in which F is well-defined and is equal to 0. We call the interval $B(n)$ a *natural support* of F. 'Usually' there will be only one such interval $B(n)$. However the polynomial factor $P(n,k)$ in F may have isolated zeros which may create several such supports $B(n)$. Such cases will be ruled out by the conditions that we are now formulating, which roughly require that the support be surrounded by zones of zero values.

Definition. *An admissible hypergeometric term $F(n,k)$ is one in which, for all sufficiently large n there is a natural support $B(n)$ such that $B(n)$ is compact and*

$$B(n) \subseteq B(n+1) \subseteq B(n+2) \subseteq \cdots \qquad (n > n_0)$$

and such that the intervals of zero values which surround $B(n)$ satisfy

$$\beta(n-j) \geq b(n) + I \qquad \text{and} \qquad \alpha(n-j) \leq a(n) - I \tag{3.3.1}$$

for $0 \leq j \leq J$ and $n > n_0$, where I and J are the orders of a k-free recurrence that F satisfies.

Example B. The function $F(n,k) = \binom{n}{k}(2n-2k)!/(2n+2k)!$ has a natural support $B(n) = [0,n]$. However the function ceases to be well-defined immediately above $b(n)$, and does not have the buffer zone of zero values that (3.3.1) requires. Hence this F is not admissible, we will not find a homogeneous recurrence satisfied by $f(n) = \sum_{k=0}^{n} F(n,k)$ by the methods of this section. In the following section we note that such sums always satisfy inhomogeneous recurrences that are easy to find from the k-free recurrence of the summand.

The function $F(n,k) = \binom{n}{k}(2n+2k)!/(2n-2k)!$, which looks similar, is quite different. It is also supported on $[0,n]$, and furthermore it vanishes on $[-n,0)$ and on (n,∞). Hence it is admissible and our theory will find a recurrence for the sums $f(n) = \sum_{k=0}^{n} F(n,k)$.

Definition. *Let F be an admissible hypergeometric term. Then the hypergeometric polynomials associated with F are the power series*

$$f_n(x) = \sum_{k \in B(n)} F(n,k)x^k \qquad (n > n_0).$$

Our goal now is to find a recurrence relation that is satisfied by the $\{f_n(x)\}$. We return to the k-free recurrence (3.1.2) that F satisfies, we multiply it by x^k and sum from $k = a(n)$ to $k = b(n) + I$. This yields

$$\sum_{j=0}^{J}\sum_{i=0}^{I} \alpha_{i,j}(n)x^i \sum_{k=a(n)}^{b(n)+I} F(n-j,k-i)x^{k-i} = \sum_{j=0}^{J}\sum_{i=0}^{I} \alpha_{i,j}(n)x^i \sum_{m=a(n)-i}^{b(n)+I-i} F(n-j,m)x^m.$$

The summand $F(n-j,m)$ in the innermost sum vanishes for values of m that lie outside of the support interval $B(n-j) = [a(n-j), b(n-j)]$, for suppose that $a(n) - i \leq m < a(n-j)$. Then $m \geq a(n) - I \geq \alpha(n-j)$, by (3.3.1), so m lies in the zone of zero values of F that surrounds $B(n-j)$. Similarly, suppose that $b(n-j) < m \leq b(n) + I - i$. Then $m \leq b(n) + I - i \leq b(n) + I \leq \beta(n-j)$, again by (3.3.1).

Hence the innermost sum is exactly $f_{n-j}(x)$ for every $i = 0, 1, \ldots, I$, and we have found that

$$\sum_{j=0}^{J} \alpha_j(n,x) f_{n-j}(x) = 0 \qquad (n > n_0), \tag{3.3.2}$$

in which the coefficients are

$$\alpha_j(n,x) = \sum_{i=0}^{I} \alpha_{i,j}(n)x^i \qquad (j = 0, 1, \ldots, J). \tag{3.3.3}$$

This recurrence is nontrivial because (3.1.2) was.

Theorem 3.2B. *Let $F(n,k)$ be an admissible hypergeometric term. Then the associated hypergeometric series $f_n(x) = \sum_{k \in B(n)} F(n,k)x^k$ satisfy the nontrivial recurrence (3.3.2).* $\square$

Finally we will obtain a nontrivial recurrence for the *sums* $f(n) = \sum_{k \in B(n)} F(n,k)$. In (3.3.2), let $(x-1)^\rho$ be the highest power of $x - 1$ that divides all of the coefficients $\alpha_{i,j}(n,x)$ $(j = 0, \ldots, J)$, and then divide through (3.3.2) by $(x-1)^\rho$. Thus we can assume w.l.o.g. that at least one of the coefficients $\alpha_j(n,x)$ does not vanish at $x = 1$. Now let $x = 1$ in (3.3.2) to obtain a nontrivial recurrence for the sums.

Theorem 3.2C. *Let $F(n,k)$ be an admissible hypergeometric term. Then the associated sums $f(n) = \sum_{k\in B(n)} F(n,k)$ satisfy a nontrivial recurrence*

$$c_0(n)f(n) + c_1(n)f(n-1) + \cdots + c_J(n)f(n-J) = 0 \qquad (n > n_0),$$

whose coefficients are polynomials in n.

This recurrence is nontrivial because the coefficients are given by $c_j = \alpha_j(n,1)$ $(j = 0,1,\ldots,J)$ and not all of the α's vanish at $x = 1$. □

3.4 Non-standard boundary conditions.

In the previous section we dealt with the case in which the summation index runs over all values of k for which the summand is nonzero, and furthermore in which that support lies well inside a larger set in which the summand remains well defined and vanishes. In such cases the limits of the sum are determined by the summand itself.

Often, of course, one is interested in sums in which the limits are not the *natural* ones referred to above. For instance, we might be interested in proving the assertion that

$$f(n) = \sum_{k=0}^{n} \frac{(2n-2k)!(2k)!}{k!(k+1)!(n-k)!^2} = \binom{2n+1}{n}. \tag{3.4.1}$$

In this case the support of the summand is indeed the same as the interval of summation, but the support is not nested inside a larger set in which the summand is well defined and vanishes. This causes no essential difficulty. It simply results in an *inhomogeneous* recurrence relation for the sum, instead of the homogeneous ones that we found in the previous section.

Example C. In the case of (3.4.1), the summand $F(n,k)$ satisfies the k-free recurrence

$$16(n-1)F(n-2,k-1) - 2(2n+1)F(n-1,k) - 2(2n-1)F(n-1,k-1) + (n+1)F(n,k) = 0, \tag{3.4.2}$$

as guaranteed by Theorem 3.1, for $1 \le k \le n$, $n \ge 2$, but not for $k = 0$ because the term $F(n-2,k-1)$ is undefined when $k = 0$. To find a recurrence for $f(n)$ of (3.4.1), we sum (3.4.2) only over $1 \le k \le n$. This yields, after some rearrangement, the recurrence

$$16(n-1)f(n-2) - 8nf(n-1) + (n+1)f(n) = -\frac{2}{n}\binom{2n-2}{n-1} \qquad (n \ge 2), \tag{3.4.3}$$

for the sums $f(n)$, which certainly proves the evaluation (3.4.1) since the claimed right hand side trivially satisfies (3.4.3). □

Example D. For another example, consider the pretty identity

$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-k} = 2^n \qquad (n \ge 0), \tag{3.4.4}$$

of [GKP], in which the upper limit of summation is not a "natural" one, in the sense of the preceding section. If $F(n,k)$ denotes the summand, then it satisfies the k-free recurrence $2F(n,k) - F(n,k-1) - 2F(n-1,k) = 0$ for $n,k \ge 1$. If $f(n)$ is the sum on the left side of (3.4.4), then by summing the k-free recurrence from $k = 0$ to

n we find that $2f(n)-(f(n)-F(n,n))-2(f(n-1)+F(n-1,n))=0$, which simplifies to $f(n)-2f(n-1)=0$, and the result follows. $\square$

The k-free recurrence for the summand will result in a proof of a claimed identity as long as the interval of summation is of the form $[rn+s, un+v]$ and the summand is well-defined throughout the range. Indeed, suppose F is a proper-hypergeometric term, and that we are interested in the sums

$$f(n)=\sum_{k=rn+s}^{un+v} F(n,k),$$

where r,s,u,v are fixed integers. Then by summing (3.2.1) on k we obtain the recurrence

$$a_0(n)f(n)+\cdots+a_J(n)f(n-J)=G(n,un+v)-G(n,rn+s-1)+\gamma(n), \tag{3.4.4}$$

where

$$\gamma(n)=\sum_{j=0}^{J} a_j(n)\Bigg\{\sum_{(n-j)r+s\le k<rn+s} - \sum_{(n-j)u+v<k\le nu+v}\Bigg\}F(n,k). \tag{3.4.5}$$

Since only a fixed number of terms are in $\gamma(n)$, in any particular case we can calculate it explicitly and then check that (3.4.4) is satisfied, to prove a claimed identity.

4. The r-variable case

4.1 The recurrence in the case of r variables.

We generalize the results of the previous section to r summation indices, with a view to finding recurrences that are satisfied by sums of the form

$$f_n(\mathbf{x})=\sum_{k_1,\ldots,k_r} F(n,k_1,k_2,\ldots,k_r)x_1^{k_1}\cdots x_r^{k_r} \tag{4.1.1}$$

for integer n, where $r\ge 1$ and the summand F is a proper-hypergeometric term.

The form of F in this case is

$$F(n,\mathbf{k})=P(n,\mathbf{k})\frac{\prod_{s=1}^{p}(a_sn+\mathbf{b}_s\cdot\mathbf{k}+c_s)!}{\prod_{s=1}^{q}(u_sn+\mathbf{v}_s\cdot\mathbf{k}+w_s)!}\mathbf{z}^{\mathbf{k}} \tag{4.1.2}$$

where P is a polynomial, the a's, u's, $\mathbf{b}$'s and $\mathbf{v}$'s are integers that contain no additional parameters, and the c's and w's are integers that may involve unspecified parameters.

Define, for integer u,

$$\Lambda_u(a)=\begin{cases}1, & \text{if } u=0;\\ (a+1)^{\overline{u}}, & \text{if } u>0;\\ 1/(a+1-|u|)^{\underline{|u|}}, & \text{if } u<0.\end{cases}$$

This Λ_u is 'really' just $(u+a)!/a!$, rewritten to emphasize that it is a polynomial in a of degree u, if the integer u is ≥ 0, or the reciprocal of a polynomial of degree $|u|$, if $u<0$.

We will write x^+ for $\max(x,0)$. Bold face letters will denote r-vectors, so that, e.g., $\mathbf{k}=(k_1,\ldots,k_r)$, and (4.1.1) may be rewritten as $f_n(\mathbf{x})=\sum_{\mathbf{k}}F(n,\mathbf{k})\mathbf{x}^{\mathbf{k}}$. An inequality between bold quantities, such as $\mathbf{k}\ge\mathbf{0}$, will mean that the inequality holds between all of their components.

We will now show that the term F itself satisfies a k-free recurrence relation, of the form

$$\sum_{0\le j\le J}\sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{I}}\alpha(\mathbf{i},j,n)F(n-j,\mathbf{k}-\mathbf{i})=0, \tag{4.1.3}$$

where the α's are polynomials in n, and we will find explicit bounds for I and J.

Fix a positive integer n and a point $\mathbf{k}$ such that $F(n,\mathbf{k})$ is well defined and nonzero. Then there is a neighborhood $\mathcal{N}(\mathbf{k})$ in, say, the complex r-dimensional space of $\mathbf{k}$, throughout which $F(n,\mathbf{k})$ remains well-defined and nonzero. We divide equation (4.1.3) by $\hat{F}(n,\mathbf{k})$, which is everything in F except for the polynomial factor $P(n,\mathbf{k})$, to obtain an equation in rational functions,

$$W(\mathbf{k})\stackrel{\text{def}}{=}\sum_{0\le j\le J}\sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{I}}\alpha(\mathbf{i},j,n)\left\{\frac{F(n-j,\mathbf{k}-\mathbf{i})}{\hat{F}(n,\mathbf{k})}\right\}=0. \tag{4.1.4}$$

We propose to find a nontrivial set of α's for which the quantity $W(\mathbf{k})$ vanishes identically in $\mathbf{k}$. To do that we will first put $W(\mathbf{k})$, which is a sum of rational functions of $\mathbf{k}$, over a common denominator. Next we will show that if I and J are large enough, then we have more α's at our disposal than conditions that they must satisfy, so the α's will exist.

The conditions on the α's are that the coefficients of all monomials $\mathbf{k}^{\mathbf{a}}$ in the numerator of

$$\begin{aligned}W(\mathbf{k})&=\sum_{\mathbf{i},j}\alpha(\mathbf{i},j,n)\frac{F(n-j,\mathbf{k}-\mathbf{i})}{\hat{F}(n,\mathbf{k})}\\&=\sum_{\mathbf{i},j}\alpha(\mathbf{i},j,n)\frac{\prod_{s=1}^{p}\Lambda_{-ja_s-\mathbf{b}_s\cdot\mathbf{i}}(a_sn+\mathbf{b}_s\cdot\mathbf{k}+c_s)}{\prod_{s=1}^{q}\Lambda_{-ju_s-\mathbf{v}_s\cdot\mathbf{i}}(u_sn+\mathbf{v}_s\cdot\mathbf{k}+w_s)}P(n-j,\mathbf{k}-\mathbf{i})\mathbf{z}^{-\mathbf{i}}\end{aligned} \tag{4.1.5}$$

must vanish.

We want to put this expression for $W(\mathbf{k})$ over a common denominator, and then consider the degrees of the numerator and the denominator polynomials in $\mathbf{k}$.

A common denominator for $W(\mathbf{k})$ is (note that we can ignore $\mathbf{z}^{-\mathbf{i}}$, which does not involve $\mathbf{k}$)

$$\prod_{s=1}^{p}\Lambda_{\rho(s)}(a_sn+\mathbf{b}_s\cdot\mathbf{k}+c_s)\prod_{s=1}^{q}\Lambda_{\sigma(s)}(u_sn+\mathbf{v}_s\cdot\mathbf{k}+w_s)$$

where

$$\rho(s)=\max\{0,\max_{j,\mathbf{i}}\{ja_s+\mathbf{b}_s\cdot\mathbf{i}\}\}\le Ja_s^{+}+I(\mathbf{b}_s)^{+}\cdot\mathbf{1},$$

$$\sigma(s)=\max\{0,\max_{j,\mathbf{i}}\{-ju_s-\mathbf{v}_s\cdot\mathbf{i}\}\}\le J(-u_s)^{+}+I(-\mathbf{v}_s)^{+}\cdot\mathbf{1},$$

and $\mathbf{1}$ is a vector of r 1's. This common denominator is a polynomial in $\mathbf{k}$ of degree

$$\Delta=I\left\{\sum_{s=1}^{p}(\mathbf{b}_s)^{+}\cdot\mathbf{1}+\sum_{s=1}^{q}(-\mathbf{v})_s^{+}\cdot\mathbf{1}\right\}+J\left\{\sum_{s=1}^{p}a_s^{+}+\sum_{s=1}^{q}(-u_s)^{+}\right\}.$$

Next we consider a single term in $W(\mathbf{k})$, say the term $(\mathbf{i},\,j)$ in the last member of (4.1.4). The degree of the numerator of that single term, as a polynomial in $\mathbf{k}$, is

$$\nu_{\mathbf{i},j}=\sum_{s=1}^{p}(-ja_s-\mathbf{i}\cdot\mathbf{b}_s)^{+}+\sum_{s=1}^{q}(ju_s+\mathbf{i}\cdot\mathbf{v}_s)^{+}+\deg(P),$$

where $\deg(P)$ is the degree of the polynomial P, and the degree of its denominator is

$$\delta_{\mathbf{i},j} = \sum_{s=1}^{p}(ja_s + \mathbf{i}\cdot\mathbf{b}_s)^+ + \sum_{s=1}^{q}(-ju_s - \mathbf{i}\cdot\mathbf{v}_s)^+.$$

Therefore when that single term is put over the common denominator it will contribute to the common numerator a polynomial in $\mathbf{k}$ of degree at most

$$\begin{aligned}
\Delta + &\max_{\mathbf{i},j}(\nu_{\mathbf{i},j} - \delta_{\mathbf{i},j}) \\
&\le \Delta + \max_{\mathbf{i},j}\left\{-j\sum_{s=1}^{p} a_s - \sum_{s=1}^{p}(\mathbf{i}\cdot\mathbf{b}_s) + j\sum_{s=1}^{q} u_s + \sum_{s=1}^{q}(\mathbf{i}\cdot\mathbf{v}_s)\right\} + \deg(P) \\
&\le \Delta + J\left\{(-\sum_{s=1}^{p} a_s)^+ + (\sum_{s=1}^{q} u_s)^+\right\} + I\left\{\sum_{s=1}^{p}(-\mathbf{b}_s)^+\cdot\mathbf{1} + \sum_{s=1}^{q}(\mathbf{v}_s)^+\cdot\mathbf{1}\right\} + \deg(P) \\
&= \left\{\sum_{s=1}^{p}\sum_{r'=1}^{r}|(\mathbf{b}_s)_{r'}| + \sum_{s=1}^{q}\sum_{r'=1}^{r}|(\mathbf{v}_s)_{r'}|\right\}I + \\
&\qquad\qquad \left\{\sum_{s=1}^{p} a_s^+ + (-\sum_{s=1}^{p} a_s)^+ + (\sum_{s=1}^{q} u_s)^+ + \sum_{s=1}^{q}(-u_s)^+\right\}J + \deg(P) \\
&\overset{\text{def}}{=} \beta I + \gamma J + \deg(\mathrm{P}), \text{ say.}
\end{aligned}$$

The number of monomials in a polynomial of degree $\beta I + \gamma J + \deg(P)$ in r variables is $\binom{r+\beta I+\gamma J+\deg(P)}{r}$, and the coefficients of all of these monomials must vanish. The number of available coefficients $\alpha(\mathbf{i}, j, n)$ is $(J+1)(I+1)^r$. Consequently we will be able to find a nontrivial set of α's if

$$(J+1)(I+1)^r \ge \binom{r+\beta I+\gamma J+\deg(P)}{r} + 1.$$

The question is this: how large must J, the order of the recurrence, be in order to guarantee the existence of some sufficiently large I for which the above inequality is true?

If J is fixed, then for I large we have

$$\binom{r+\beta I+\gamma J+\deg(P)}{r} \sim \frac{\beta^r I^r}{r!}.$$

Hence $(I+1)^r(J+1)$ will surely be larger than this, for all large enough I, if $J > \beta^r/r! - 1$, i.e., if

$$J \ge \left\lfloor \frac{1}{r!}\left\{\sum_{s=1}^{p}\sum_{r'=1}^{r}|(\mathbf{b}_s)_{r'}| + \sum_{s=1}^{q}\sum_{r'=1}^{r}|(\mathbf{v}_s)_{r'}|\right\}^r \right\rfloor. \tag{4.1.6}$$

Theorem 4.1. *Every proper-hypergeometric term in r variables satisfies a nontrivial k-free recurrence relation. Indeed there exist I, J and polynomials $\alpha(\mathbf{i}, j, n)$, not all zero, such that (4.1.3) holds at every point $(n_0, \mathbf{k}_0) \in \mathbb{Z}^{r+1}$ for which $F(n_0, \mathbf{k}_0) \ne 0$ and all of the values $F(n_0 - j, \mathbf{k}_0 - \mathbf{i})$ that occur in (4.1.3) are well-defined. Furthermore there is such a recurrence in which $J = J^*$, where J^* is the right member of (4.1.6) above.* $\square$

4.2 Certification of multivariable identities.

In this section we will develop the r-variate analogs of the certification theorems of section 3.2 above. These will all be consequences of theorem 4.1. The hypotheses of that theorem will suffice for our results that apply to the summand F itself, but we will, as before, need stronger hypotheses to deal with recurrences for definite sums of values of the summand.

The k-free recurrence (4.1.3) that is satisfied by the summand F can be written in operator form. We let N be the operator that shifts (down) the variable n: $Nf(n) = f(n-1)$. Further, for each $i = 1, \ldots, r$ we let K_i be the operator that shifts the variable k_i: $K_i f(\mathbf{k}) = f(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$, and we will use Δ_i for the forward difference operator on the ith coordinate.

Then (4.1.3) is equivalent to an assertion

$$H(N, n, K_1, \ldots, K_r)F(n, \mathbf{k}) = 0,$$

where H is a polynomial in its arguments and does not involve $\mathbf{k}$. We can expand H in a Taylor's series about $\mathbf{K} = \mathbf{1}$, to obtain

$$H(N, n, \mathbf{K}) = H(N, n, \mathbf{1}) + \sum_{i=1}^{r}(K_i - 1)V_i(N, n, \mathbf{K})$$

in which the V_i are polynomials in their arguments. It follows that

$$\begin{aligned} H(N, n, \mathbf{1})F(n, \mathbf{k}) &= \sum_{i=1}^{r}(1 - K_i)V_i(N, n, \mathbf{k})F(n, \mathbf{k}) \\ &\stackrel{\text{def}}{=} \sum_{i=1}^{r}\big\{G_i(n, k_1, \ldots, k_i, \ldots, k_r) - G_i(n, k_1, \ldots, k_i - 1, \ldots, k_r)\big\}. \end{aligned}$$

Thus we have the following generalization of theorem 3.2A.

Theorem 4.2A. *Let F be a proper-hypergeometric term. Then there are a positive integer J, polynomials $a_0(n), \ldots, a_J(n)$ and hypergeometric functions $G_1, \ldots, G_r$ such that for every $(n, \mathbf{k}) \in \mathbb{N}^{r+1}$ at which $F \neq 0$ and F is well-defined at all of the arguments that appear in (4.1.3) we have*

$$a_0(n)F(n, \mathbf{k}) + \cdots + a_J(n)F(n - J, \mathbf{k}) = \sum_{i=1}^{r}\Delta_i G_i(n, \mathbf{k}).$$

Moreover this recurrence is nontrivial, and each $G_i(n, \mathbf{k})$ is of the form $R_i(n, \mathbf{k})F(n, \mathbf{k})$, where the R's are rational functions of their arguments. □

Next we will pass from theorems about the summand F to theorems about the sums $\sum_{\mathbf{k}} F(n, \mathbf{k})$, and we will now formulate conditions that will permit this, for standard boundary conditions.

From the form (4.1.2) of the proper-hypergeometric term F in the multivariate case, we define two convex polyhedra in $\mathbb{E}^r$. The first of these is $\mathcal{S}(n)$, the *support* of $F(n, \cdot)$, which is

$$\mathcal{S}(n) = \cap_{s=1}^{q}\{\mathbf{k} : u_s n + \mathbf{v}_s \cdot \mathbf{k} + w_s \geq 0\}.$$

The second is the set $\mathcal{W}(n)$ on which F is well-defined, and it is

$$\mathcal{W}(n) = \cap_{s=1}^{q}\{\mathbf{k} : a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s \geq 0\}.$$

If $\mathbf{k} \in \mathcal{W}(n) \backslash \mathcal{S}(n)$ then $F(n, \mathbf{k}) = 0$.

Next we define a set $T(n)$, which is an I-neighborhood of $\mathcal{S}(n)$. It is

$$T(n) = \{\mathbf{k} : \exists \mathbf{k}' \in \mathcal{S}(n) \ s.t.\ ||\mathbf{k} - \mathbf{k}'||_\infty \leq I\}.$$

Definition. *Let F be a proper-hypergeometric term that has a k-free recurrence of orders (I, J). Then F is an admissible hypergeometric term if for all $n > n_0$ we have*

H1 *(The sums are terminating) $\mathcal{S}(n)$ is compact and $\mathcal{S}(n) \subseteq \mathcal{W}(n)$, and*

H2 *$\mathcal{S}(n) \subseteq \mathcal{S}(n+1) \subseteq \cdots$, and*

H3 *(Existence of zone of 0's outside of $\mathcal{S}(n)$) For each $j = 0, \ldots, J$, and for all $\mathbf{i}$ s.t. $||\mathbf{i}||_\infty \le I$ we have $T(n) + \mathbf{i} \subseteq W(n-j)$.*

The multisums that we are now considering are those in which the summation indices $\mathbf{k}$ run over all lattice points in $\mathcal{S}(n)$, i.e., the case of standard boundary conditions.

We define the hypergeometric polynomials

$$f_n(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{S}(n)} F(n, \mathbf{k}) \mathbf{x}^{\mathbf{k}}, \tag{4.2.1}$$

and we seek a recurrence that they satisfy.

To find this we multiply (4.1.3) by $\mathbf{x}^{\mathbf{k}}$ and sum over $\mathbf{k} \in T(n)$. This gives

$$\begin{aligned} 0 &= \sum_{0 \le j \le J} \sum_{\mathbf{0} \le \mathbf{i} \le \mathbf{I}} \alpha(\mathbf{i}, j, n) \sum_{\mathbf{k} \in T(n)} F(n-j, \mathbf{k} - \mathbf{i}) \mathbf{x}^{\mathbf{k}} \\ &= \sum_{0 \le j \le J} \sum_{\mathbf{0} \le \mathbf{i} \le \mathbf{I}} \alpha(\mathbf{i}, j, n) \mathbf{x}^{\mathbf{i}} \sum_{\mathbf{k} \in T(n)} F(n-j, \mathbf{k} - \mathbf{i}) \mathbf{x}^{\mathbf{k} - \mathbf{i}} \\ &= \sum_{0 \le j \le J} \sum_{\mathbf{0} \le \mathbf{i} \le \mathbf{I}} \alpha(\mathbf{i}, j, n) \mathbf{x}^{\mathbf{i}} \sum_{\mathbf{k}' \in T(n) - \mathbf{i}} F(n-j, \mathbf{k}') \mathbf{x}^{\mathbf{k}'}. \end{aligned}$$

Consider the innermost sum for some fixed $\mathbf{i}$. We claim that it is exactly the hypergeometric series $f_{n-j}(\mathbf{x})$ of (4.2.1).

First, every $\mathbf{y} \in \mathcal{S}(n-j)$ occurs as a $\mathbf{k}'$ in this sum. Indeed, every $\mathbf{y}$ in the larger set $\mathcal{S}(n)$ occurs, since

$$\mathbf{y} \in \mathcal{S}(n),\ ||(\mathbf{y} + \mathbf{i}) - \mathbf{y}||_\infty \le I \quad \Rightarrow \quad \mathbf{y} + \mathbf{i} \in T(n) \quad \Rightarrow \quad \mathbf{y} \in T(n) - \mathbf{i}.$$

Finally, if we fix some $\mathbf{k}' \in T(n) - \mathbf{i} \backslash \mathcal{S}(n-j)$ then we claim that $F(n-j, \mathbf{k}') = 0$, i.e., that $\mathbf{k}' \in \mathcal{W}(n-j) \backslash \mathcal{S}(n-j)$. For, by **H3**, $\mathbf{k}' \in \mathcal{W}(n-j)$. But $\mathbf{k}' \notin \mathcal{S}(n-j)$, which completes the proof of the following result.

Theorem 4.2B. *Let $F(n, \mathbf{k})$ be an admissible hypergeometric term. Then the hypergeometric polynomials $\{f_n(x)\}_{n > n_0}$ satisfy a nontrivial recurrence relation*

$$\sum_{j=0}^{J} \alpha_j(n, \mathbf{x}) f_{n-j}(\mathbf{x}) = 0 \tag{4.2.2}$$

in which the coefficients

$$\alpha_j(n, \mathbf{x}) = \sum_{0 \le i_1, \ldots, i_r \le I} \alpha(\mathbf{i}, j, n) \mathbf{x}^{\mathbf{i}}$$

are polynomials in n and $x_1, \ldots, x_r$. $\square$

Now divide through (4.2.2) by the highest power of $(x_r - 1)$ that divides into all of the α_j, then let $x_r = 1$. The result is still a nontrivial recurrence. Repeat with $x_{r-1}, \ldots, x_1$ the operation of dividing out the highest power and setting the variable equal to 1 to obtain a nontrivial recurrence for the sums.

Theorem 4.2C. *Let F be an admissible hypergeometric term and let*

$$f(n) = \sum_{\mathbf{k} \in \mathcal{S}(n)} F(n, \mathbf{k}) \qquad (n > n_0).$$

Then $\{f(n)\}$ satisfies a nontrivial linear, homogeneous recurrence relation of order J, with coefficients that are polynomials in n. □

The treatment of nonstandard boundary conditions proceeds as in the case of a single summation variable, but the boundary terms "snowball" so the application will become rapidly more complicated.

5. Generalization to q-sums and q-multisums

5.1 q-identities.

In this section we will prove that q-proper-hypergeometric terms also satisfy recurrence relations with polynomial coefficients, and we will obtain quite explicit bounds for the order of such a recurrrence. This will be done first for the case of a single variable of summation, and then for multi-q-hypergeometric terms as well. As indicated at the end of sub-section 1.6 above, this analysis, via the isomorphism stated there, also applies to integration and to taking the constant term.

We will prove this result for q-hypergeometric terms of the form

$$F(n, k) = \frac{\prod_s Q(a_s n + b_s k, c_s)}{\prod_s Q(u_s n + v_s k, w_s)} q^{an^2+bnk+ck^2+dk+en} \xi^k \tag{5.1.1}$$

where

$$Q(m, c) = (1 - cq)(1 - cq^2) \cdots (1 - cq^m). \tag{5.1.2}$$

Our hypotheses about the parameters etc. will be the same as in the preceding section.

As before, we seek I, J such that for some nontrivial α's we have

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \alpha(i, j; n) F(n - j, k - i) = 0.$$

Theorem 5.1. *Let F be a q-hypergeometric term of the form (5.1.1). Then F satisfies a k-free recurrence whose order J is at most $\sum_s b_s^2 + \sum_s v_s^2 + 2|c|$.*

Proof. As before, if $F(n, k) \neq 0$ we divide by it to get

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \alpha(i, j; n) \frac{F(n - j, k - i)}{F(n, k)} = 0. \tag{5.1.3}$$

Next we will find a common denominator for the $(I+1)(J+1)$ ratios $F(n-j, k-i)/F(n, k)$ that appear. Then we will express each of those ratios as a certain numerator divided by that common denominator. The sum of all of the numerators will then be required to vanish identically in k. In the previous section, to achieve this goal we equated to zero the coefficient of each power of k that appeared in the common numerator. In this section we will equate to zero the coefficients of each power of q^k that appears in the common numerator. The variable k will not appear in that common numerator in any form other than powers of q^k, so these conditions will determine the α's that we seek.

We define, by analogy with the rising factorial, for integers α, β and a complex number c,

$$\Lambda_\beta(\alpha; c) = \frac{Q(\alpha+\beta, c)}{Q(\alpha, c)} = \begin{cases} (1-cq^{\alpha+1})\cdots(1-cq^{\alpha+\beta}), & \text{if } \beta > 0; \\ 1, & \text{if } \beta = 0; \\ \{(1-cq^{\alpha+\beta+1})\cdots(1-cq^{\alpha})\}^{-1} & \text{if } \beta < 0. \end{cases} \tag{5.1.4}$$

Now we have

$$\frac{F(n-j, k-i)}{F(n,k)} = \frac{\prod_s \Lambda_{-ja_s - ib_s}(a_s n + b_s k; c_s)}{\prod_s \Lambda_{-ju_s - iv_s}(u_s n + v_s k; w_s)} \frac{q^{\phi(i,j)n+\psi(i,j)}}{q^{k(bj+2ci)}}. \tag{5.1.5}$$

Observe that certain factors may appear in the *denominator* of (5.1.5) but actually contribute to the *numerator* of the rational function of q^k that (5.1.5) really is. When we wish to make this distinction we will speak of the *apparent* denominator of (5.1.5) or of its *true* denominator.

We now want to calculate the degree ν_{ij} of the true numerator of (5.1.5), regarded as a polynomial in $t = q^k$, and the degree δ_{ij} of its true denominator. For this purpose there are four estimates to make: in estimate NN, we will find the contribution of a typical factor of the product in the apparent **N**umerator of (5.1.5) to the true **N**umerator. In estimate ND we compute the contribution of a factor in the apparent **N**umerator to the true **D**enominator, and similarly for the estimates DN and DD.

Estimate ND: Consider a typical factor of the product in the apparent numerator of (5.1.5),

$$\Lambda_{-ja-ib}(an+bk; c) = (1-cq^{an+bk+1})\cdots(1-cq^{an+bk+(-ja-ib)}),$$

in which $ja + ib < 0$. If we let $t = q^k$, then this factor is a polynomial in t^b of degree $|ja+ib|$. If $b > 0$ then this factor does not contribute to the true denominator of (5.1.5). If $b < 0$ then this factor is of the form

$$\text{poly of degree } |ja+ib| \text{ in } t^{-|b|} = t^{-|b(ja+ib)|}(\text{poly of degree } |ja+ib| \text{ in } t^{|b|}).$$

Hence a factor in the apparent numerator with $ja + ib < 0$ contributes a factor $t^{|(-b)^+(ja+ib)|}$ to the true denominator of (5.1.5).

If, on the other hand $ja + ib > 0$ then the factor in question is the reciprocal of a polynomial in t^b of degree $ja + ib$. Hence if $b > 0$ we have a contribution of $b(ja + ib)$ to the degree of the true denominator, and if $b < 0$, a contribution of $|b|(ja + ib)$.

To summarize estimate ND, then, a factor in the apparent numerator of (5.1.5) contributes a polynomial in $t = q^k$ of degree $(-b)^+|ja+ib|$, if $ja+ib < 0$, or of degree $|b|(ja+ib)$, if $ja+ib > 0$, to the true denominator of (5.1.5). We can write this in a single formula as $|b_s|(ja_s+ib_s)^+ + (-b_s)^+(-ja_s-ib_s)^+$.

Estimate NN: If $ja_s + ib_s > 0$ then this same factor of the product in the apparent numerator is the reciprocal of a polynomial of degree $ja_s + ib_s$ in t^{b_s}. If $b_s > 0$, it makes no contribution to the true numerator. If $b_s < 0$ we would multiply top and bottom by $t^{|b_s|(ja_s+ib_s)}$ so as to obtain a polynomial in $t^{|b_s|}$ in the true denominator. This process would, in that case, yield a polynomial of degree $|b_s|(ja_s+ib_s)$ in the true numerator.

Hence if $ja_s + ib_s > 0$ we get a factor in the true numerator of degree $(-b_s)^+(ja_s + ib_s)$. Similarly if $ja_s + ib_s < 0$ the contribution to the degree of the true numerator is $|b_s(ja_s + ib_s)|$.

To summarize estimate NN, a factor in the apparent numerator of (5.1.5) contributes a polynomial in $t = q^k$ of degree $(-b)^+(ja+ib)$, if $ja+ib > 0$, or of degree $|b(ja+ib)|$, if $ja+ib < 0$, to the true numerator of (5.1.5).

Estimate DN: Omitting the details, the contribution of a factor of the product in the apparent denominator of (5.1.5) to the true numerator is of degree

$$|v_s|(ju_s+iv_s)^+ + (-v_s)^+(-ju_s-iv_s)^+.$$

Estimate DD: The contribution of a factor of the product in the apparent denominator of (5.1.5) to the true denominator is of degree $(-v_s)^+(ju_s+iv_s)^+ + |v_s|(-ju_s-iv_s)^+$.

The above estimates account for the product symbols that appear in (5.1.5). The factor $q^{k(jb+2ic)}$ in (5.1.5) contributes a factor of degree $(jb+2ic)^+$ to the true denominator, and of degree $(-jb-2ic)^+$ to the true numerator.

We can now write down the degrees of the true numerator and denominator of (5.1.5). The degree of the numerator is

$$\begin{aligned}\nu_{ij} = &\sum_s \{(-b_s)^+(ja_s+ib_s)^+ + |b_s|(-ja_s-ib_s)^+\} \\ &\quad + \sum_s \{|v_s|(ju_s+iv_s)^+ + (-v_s)^+(-ju_s-iv_s)^+\} + (-jb-2ic)^+,\end{aligned} \tag{5.1.6}$$

and the degree of the denominator is

$$\begin{aligned}\delta_{ij} = &\sum_s \{(-b_s)^+(-ja_s-ib_s)^+ + |b_s|(ja_s+ib_s)^+\} \\ &\quad + \sum_s \{|v_s|(-ju_s-iv_s)^+ + (-v_s)^+(ju_s+iv_s)^+\} + (jb+2ic)^+.\end{aligned} \tag{5.1.7}$$

What we want is the degree of a common denominator of all of the expressions (5.1.5) as i and j vary. Among the polynomials that contributed to the foregoing estimates of the denominator degree δ_{ij}, there were just two kinds of polynomials. First, some were pure powers of t. Evidently all powers of t that occur are divisors of t raised to the highest power of t that occurs.

Second, there were partial products of consecutive terms, of the form $(1-d_1t)\cdots(1-d_mt^m)$, in which the 'm' appears in our degree estimate. These partial products have the property that if one of them has a higher m than another then the second one divides the first one, as long as we maintain the same initial coefficients $\{d_i\}$ in both. So once again, the maximum degree that occurs can be used as the degree of a single polynomial that is divisible by all of the polynomials of this kind that occur.

The conclusion is that if we maximize every term in the estimate (5.1.7) for the denominator degree δ_{ij}, then we will have an upper bound for the degree of a common denominator for all of the terms that occur in (5.1.3). Thus we need an estimate for the maximum of (5.1.7) over all $0 \le i \le I$ and all $0 \le j \le J$. These estimates are all quite trivial to make and we omit the details. The result is that there is a common denominator for all of the terms in (5.1.3), whose degree is at most

$$\begin{aligned}\Delta = I&\Big\{\sum_s b_s^2 + \sum_s v_s^2\tau(\text{“}v_s<0\text{”}) + 2c^+\Big\} \\ &+ J\Big\{\sum_s |a_sb_s|\tau(\text{“}a_s>0 \text{ or } b_s<0\text{”}) + \sum_s |u_sv_s|\tau(\text{“}u_s<0 \text{ or } v_s<0\text{”}) + b^+\Big\},\end{aligned} \tag{5.1.8}$$

where $\tau(\mathcal{P})$ is the truth value ($= 0$ or 1) of the proposition $\mathcal{P}$.

Next we will put all of the terms in (5.1.3) over the common denominator that was just discussed. We wish to estimate the degree of the common numerator. When a particular term (i, j) is put over that denominator it contributes a polynomial of degree $\Delta + \nu_{ij} - \delta_{ij}$, where these three quantities are given by (5.1.8), (5.1.6) and (5.1.7), respectively. If we compute directly from these three formulas, and we use the fact that for all a, $a^+ - (-a)^+ = a$, then we find that a great deal of simplification occurs and we have

$$\begin{aligned}\nu_{ij} + \Delta - \delta_{ij} &= \Delta + \sum_{v_s>0} v_s(ju_s + iv_s) - \sum_{b_s>0} b_s(ja_s + ib_s) - (jb + 2ic)\\ &\le \Delta + \sum_{v_s>0} v_s(J(u_s)^+ + Iv_s) + \sum_{b_s>0} b_s J(-a_s)^+ + J(-b)^+ + I(-2c)^+\\ &= I\{\sum_s b_s^2 + \sum_s v_s^2 + 2|c|\} + J\{\sum_s |a_s b_s| + \sum_s |u_s v_s| + |b|\}.\end{aligned}$$

We temporarily abbreviate the last member above by $\xi I + \eta J$. Now that we know the degree of the numerator in (5.1.3), as a polynomial in q^k, we can equate to zero the coefficient of each power of q^k that appears. The result will be $\xi I + \eta J + 1$ homogeneous linear equations in $(I + 1)(J + 1)$ unknowns α_{ij}. Hence if $(I + 1)(J + 1) \ge \xi I + \eta J + 2$ then we will have a nontrivial solution vector α, and therefore we will have found a nontrivial recurrence relation of order J whose coefficients are of degree $\le I$ in x, for the q-hypergeometric functions $\{f_n(x)\}$.

To find the lowest estimate of J, we note that $(I+1)(J+1) \ge \xi I + \eta J + 2$ surely holds for all sufficiently large I provided that $J + 1 > \xi$, i.e., provided that $J \ge J^* = \lfloor \xi \rfloor$. Hence there is certainly a recurrence of order $\sum_s b_s^2 + \sum_s v_s^2 + 2|c|$, which completes the proof of theorem 5.1. $\square$

5.2 Certification and the multi-q case.

The remaining theorems about q-identities are sufficiently similar to their non-q counterparts that in this section we will only summarize them briefly. In regard to multi-q identities we will choose the route of displaying only a very crude upper bound on the order of the claimed recurrence for the sums and for the associated polynomials. This will prove the existence of the recurrence but will give far from the tightest bounds. In particular cases, however, once one is assured that the recurrences exist, it seems best to try computationally to find such recurrences of very low orders, and work upwards slowly, since they seem to have much lower orders than one might have expected.

The definition of a q-admissible proper-hypergeometric term is identical with the one in section 4.2 above. The idea of a natural compact support buffered by a zone of 0 values is exactly what is needed here also. With that hypothesis one has immediately the q-analogs of theorems 4.2B and 4.2C, and they are identical in form to those two theorems.

Obtaining an upper bound on the order of a recurrence in the multi-q case is conceptually the same as in the earlier contexts, but the implementation would be extremely lengthy because of the complicated estimates of the degrees. So we will simplify it considerably, at the cost of some precision.

As before, we assume a linear combination of the form

$$\sum_{i,j} \alpha_{i,j}(n) F(n - j, \mathbf{k} - \mathbf{i}) = 0$$

in which F is now

$$F(n, \mathbf{k}) = \frac{\prod_{s=1}^{S} Q(a_s n + \mathbf{b}_s \cdot \mathbf{k}, c_s)}{\prod_{s=1}^{SS} Q(u_s n + \mathbf{v}_s \cdot \mathbf{k}, w_s)} q^{A(n,\mathbf{k})} \xi^{\mathbf{k}}, \tag{5.2.1}$$

where A is a quadratic form in its $r+1$ arguments, and we want to determine the α's.

After dividing by a nonzero $F(n,\mathbf{k})$, a typical term of the above, say the (i,j) term, looks like (5.1.5), namely

$$\frac{F(n-j,\mathbf{k}-\mathbf{i})}{F(n,\mathbf{k})}=\frac{\prod_s \Lambda_{-ja_s-\mathbf{i}\cdot\mathbf{b}_s}(a_sn+\mathbf{b}_s\cdot\mathbf{k};c_s)}{\prod_s \Lambda_{-ju_s-\mathbf{i}\mathbf{v}_s}(u_sn+\mathbf{v}_s\cdot\mathbf{i};w_s)}\frac{q^{A(n-j,\mathbf{k}-\mathbf{i})}}{q^{A(n,\mathbf{k})}}\xi^{-\mathbf{i}}. \tag{5.2.2}$$

The key observation is still true, namely that if $t_1<t_2$ then $\Lambda_{t_1}\backslash\Lambda_{t_2}$. Hence every denominator that occurs in (5.2.1) is a divisor of a denominator in which the Λ's have the largest possible subscripts. Now let

$$B=\max_{s,h}\{|(\mathbf{b}_s)_h|,|(\mathbf{v}_s)_h|,|a_s|,|u_s|\}+\max_{\mu,\nu}|a_{\mu,\nu}|,$$

where now $(a_{\mu,\nu})$ is the coefficient matrix of the quadratic form A that appears in (5.2.1).

The last member of (5.2.2) has the form

$$\text{const.}\frac{\prod_{\gamma_s<0}\text{poly}_{|\gamma_s|}(\mathbf{t}^{\mathbf{b}_s})\prod_{\beta_s>0}\text{poly}_{\beta_s}(\mathbf{t}^{\mathbf{v}_s})}{\prod_{\gamma_s>0}\text{poly}_{\gamma_s}(\mathbf{t}^{\mathbf{b}_s})\prod_{\beta_s<0}\text{poly}_{|\beta_s|}(\mathbf{t}^{\mathbf{v}_s})}\mathbf{t}^{-2A\mathbf{i}+c}$$

in which $\text{poly}_{\dots}$ is a polynomial of degree $\dots$ in its argument, and

$$\mathbf{t}=(t_1,\dots,t_r)=(q^{k_1},\dots,q^{k_r}),$$

$\beta_s=ju_s+\mathbf{i}\cdot\mathbf{v}_s$, $\gamma_s=ja_s+\mathbf{i}\cdot\mathbf{b}_s$, and "const." is independent of $\mathbf{t}$.

None of the poly's in the denominator has degree larger than $B(I+J+1)$, hence no exponent of any t_h in a denominator factor exceeds $B^2(I+J+1)$, in absolute value. Hence in the complete denominator polynomial, no exponent of any t_h exceeds $(S)(SS)B^2(I+J+1)$, in absolute value. If we multiply top and bottom by $(t_1\cdots t_r)^{(S)(SS)B^2(I+J+1)}$ to make all of the exponents nonnegative, then in the numerator and in the denominator polynomials none of the variables appears to a power higher than $2S(SS)B^2(I+J+1)$. Hence after putting the entire sum over a common denominator, the numerator polynomial, regarded as a polynomial in $t_1,\dots,t_r$, has no variable raised to a power higher than $\Delta=4S(SS)B^2(I+J+1)$.

The conditions on our α's are that the coefficient of each monomial in the numerator polynomial must vanish. A polynomial in r variables, the degree of each of which is $\le\Delta$, can have no more than $\binom{\Delta+r}{r}$ monomials. Hence if

$$(J+1)(I+1)^r>\binom{4S(SS)B^2(I+J+1)}{r}$$

then a nontrivial set of α's is guaranteed to exist. Now for fixed J, as $I\to\infty$, the right hand member is $\sim\text{const.}^rI^r/r!$. Hence if $J+1>\text{const.}^r/r!$, a nontrivial solution exists.

6. Examples

In this section we give a number of examples of identities and their computer-discovered proofs. These examples have a number of features in common. For one thing, they are entirely unmotivated, since the computer places no value on the reader's state of mind.[4] Second, they are quite easy to check, even though they were time-consuming for the computer to find in the first place. In each case we exhibit certain rational functions $R_1,R_2,\dots$ and we invite the reader to consider an equation of the form

$$(\text{operator acting on the index } n)F(n,..)=\Delta_{k_1}(R_1F)+\Delta_{k_2}(R_2F)+\dots+D_{x_1}(\tilde{R}_1F)+\dots, \tag{**}$$

[4]But of course they are meta-motivated by our general method.

where "operator .." is a linear difference operator in n with polynomial coefficients and the Δ's are forward difference operators with respect to the variables indicated in their subscripts, and the D's are partial differentiations.

Concerning this equation there will be two tasks for the reader: first one must check that it is true, and second one must check that it implies the identity that we are trying to prove. It must be emphasized that both of these will be trivial tasks. In order to check that $(**)$ is true one proceeds as follows:

- • Carry out the indicated operations on F and on the rational multiples of F.
- •• Divide through $(**)$ by F and cancel out factors that can be cancelled. Clear denominators.
- ••• What will remain to verify will be a *polynomial* identity in several variables that will in each case be a triviality.

In order to check that $(**)$ in each case implies the claimed identity one will simply sum (and/or integrate) both sides of $(**)$ over all of the variables that occur as subscripts of operators on the right side, and notice the telescoping that occurs, which, along with the vanishing of the summands at infinity, will complete the proof of the claimed identity.

6.1 Multi-sums.

6.1.1 An example of a proof of an explicitly evaluable double sum

The following is a computer-generated proof of a recent identity conjectured by Szondy and Varga [SzVa].

Theorem 6.1.1. *Let $F(j,m)$ be*

$$\frac{(-1)^{j+n}(2n-2m+j)!(2k+1+j)!i!(2k-2i+1)!n!(k-i-n)!}{2^{2n}j!(n-m)!(i+m)!(2k+1)!(k-i)!(m-j)!(n-m+j)!(2k-2i-2m+1+j)!}.$$

Then the double sum of F w.r.t. j and m is identically 1.

Proof. This is trivial for $n=0$, so calling the sum $a(n)$, it would be enough to prove that $a(n+1)-a(n)=0$, for every non-negative integer n. To this end, we construct two rational functions: the function $R_1(n,j,m)$, whose numerator is

$$\begin{aligned}&(-4j^2k-3nj-4ji^2+4mj-2n-j+2m-j^2-4m^2-4n^2-4ni^2-8m^2k+4mi^2\\&-4jk^2+4n^2j-4n^2m+8nm+3j^2n+2ni-2mi+2mk+8m^2i+4mk^2-4nk^2\\&+4m^2n-2nk+2ji-2jk+4j^2i-8mjn+8nmk-8nmi+8kji-8kmi+8kni\\&-4njk-12mji+4nji+12mjk)(-j/4)\end{aligned}$$

and whose denominator is $(2n-2m+j)(-1-m+j)(n+1+i)(-k+i+n)$. Further we construct the rational function $R_2(n,j,m)$ whose numerator is

$$\begin{aligned}&-(i+m)(2n-2m+j+1)(4n^2j-4n^2m+j^2n-2njk+4m^2n-6mjn-4nm\\&+6nj+4nmk-4nmi+2nji+2j+4mjk-4mji-2jk+2ji-4m^2k+4m^2i\\&+4mk-4mi-2mj+j^2)\end{aligned}$$

and whose denominator is $4(n+1-m)(n+1-m+j)(2k-2i-2m+2+j)(n+1+i)(-k+i+n)$. It is then routinely verifiable that (check!)

$$\Delta_n F = \Delta_j(R_1 F) + \Delta_m(R_2 F),$$

where the Δ's are the usual forward difference operators, and the assertion follows upon double-summing w.r.t. j and m. $\square$

6.1.2 An example of a proof of an identity of the form "double sum = single indefinite sum"

We give here a computer generated proof of the following identity of Carlitz [Ca1] (also stated by Comtet [Co], p. 172):

$$\sum_{i+j+k=n} \binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k} = \sum_{l=0}^{n} \binom{2l}{l}.$$

The computer found and proved a recurrence satisfied by the left side that is easily seen to be also satisfied by the right side, since the recurrence operator factorizes nicely into $((n+2)N-(4n+6))(N-1)$.

Theorem 6.1.2. *Let*

$$F(i,j,n) := \frac{(i+j)!(n-j)!(n-i)!}{i!^2 j!^2 (n-i-j)!^2},$$

and let $a(n)$ be its double sum w.r.t. j and i. Let N be the forward shift operator in n. The sequence $a(n)$ satisfies the recurrence

$$(4n+6+(-8-5n)N+(2+n)N^2)a(n) = 0.$$

Proof. Let Δ_i and Δ_j be the forward difference operators in i and j respectively. Define two rational functions as follows. First, $R_1(i,j)$ has for its numerator

$$\begin{aligned}(-6i^2 - 2i^2n + n^3 + 5n^2 + 4 + 8n - 4ij + nij + 4i^2j + 2i^3 - 8j - 14nj - 5n^2j \\ + 4nj^2 + 6j^2 - n^2i - ni + 2i)(n-i+1)i^2\end{aligned}$$

and its denominator is $(i+j-n-1)^2(i+j-n-2)^2(i+j)$. Next define

$$R_2(i,j) = \frac{(j-n-1)j^2(-n+i-1)(4i^2+2ij-4i-2ni+nj-4n-n^2+2j-4)}{(i+j-n-1)^2(i+j-n-2)^2(i+j)}.$$

It is then routinely verifiable that

$$(4n+6+(-8-5n)N+(2+n)N^2)F(i,j,n) = \Delta_i(R_1 F) + \Delta_j(R_2 F)$$

and the result follows by summing w.r.t. i and j. $\square$

6.1.3 An example of an identity of the form "double sum=single sum"

The following identity is due to Carlitz [Ca2] (see also [St], p. 262):

$$\sum_{\substack{i+k=m \\ j+l=n}} \binom{i+j}{i}\binom{j+k}{j}\binom{k+l}{k}\binom{i+l}{l} = \frac{(m+n+1)!}{m!n!}\sum_k \frac{1}{2k+1}\binom{m}{k}\binom{n}{k}.$$

Our algorithm found recurrences for both sides, that turned out to be identical and second order. Hence the identity follows once the two trivial initial values $n = 0, 1$ are checked. Since the right side is a single sum that is handled by [Z2] [Z3], we give only the certificates for the left.

The operator $P(N, n)$ annihilating $a(n)$, the left side (that also happened to annihilate the right side), was found to be

$$2(m+3+n)(2+m+n)^2 - (4n^2+2nm+15n+3m+14)(m+3+n)N + (5+2n)(n+2)^2N^2$$

and the "certificates" R_1 and R_2 that satisfy, (calling the summand on the left $F(n, i, j)$)

$$P(N, n)F(n, i, j) = \Delta_i(R_1F(n, i, j)) + \Delta_j(R_2F(n, i, j)),$$

and whose existence certifies that $P(N, n)$ indeed annihilates the left side $a(n)$ are,

$$R_1 = (P/D), \ R_2 = (Q/D),$$

where,

$$D := (-m+i-1)^2(i+j)(-n+j-2)^2(-n+j-1)^2,$$

and P and Q are certain polynomials that are not given here to save space, and that can be easily reproduced by the readers once we tell them that P has degrees 8 and 5 respectively in i, j, and Q has degree 6 in both i and j.

6.2 Multi-integrals.

6.2.1 The basketball numbers

The binomial coefficients $\binom{n+m}{m}$ may be defined as the number of possible "soccer games" for which the final score is $n : m$. Of course they satisfy a first order recurrence in both n and m. The basketball analog (in which one "score" is either one point or two points) is much more complicated, and turn out to satisfy a fourth order recurrence in each of n and m. The output was as follows.

Theorem 6.2.1. *Let*

$$F(x, y, m) := \frac{1}{(1-x-x^2-y-y^2)x^{m+1}y^{n+1}}$$

and let $a(m)$ be its double (contour) integral around the origin, w.r.t. x and y. Let M be the forward shift operator in m. The sequence $a(m)$ satisfies the recurrence

$$\begin{aligned}&(4(m+3)(3+n+m)(m+2+n) + 2(2mn+5n+21m+4m^2+26)(3+n+m)M+\\ &\quad (m+2)(-5m^2-54-33m-6mn-17n+n^2)M^2 - (m+3)(m+2)(34+9m+5n)M^3\\ &\qquad + 5(4+m)(m+3)(m+2)M^4)a(m) = 0.\end{aligned}$$

Proof It is routinely verifiable that

$$\begin{aligned}&(4(m+3)(3+n+m)(m+2+n) + 2(2mn+5n+21m+4m^2+26)(3+n+m)M+\\ &(m+2)(-5m^2-54-33m-6mn-17n+n^2)M^2 - (m+3)(m+2)(34+9m+5n)M^3\\ &+ 5(4+m)(m+3)(m+2)M^4)F(x, y, m)\\ &= D_x\left(\frac{\text{certain polynomial of degree 4 in } x, \text{degree 1 in } y}{(1-x-x^2-y-y^2)x^{m+4}y^{n+1}}F\right)\\ &+ D_y\left(\frac{[\text{certain polynomial of degree 2 in } x, \text{ degree 1 in } y](1+2x)}{(1-x-x^2-y-y^2)x^{m+4}y^{n}}F\right),\end{aligned}$$

and the result follows by double integrating w.r.t. x and y. $\square$

6.2.2 An identity equivalent to the Pfaff-Saalschütz identity

Gessel and Stanton noticed ([GS], see also [St], p. 192) that the celebrated Pfaff-Saalschütz identity is equivalent to the following identity.

$$\frac{(1+x)^k(1+y)^b}{(1-xy)^{k+b+1}} = \sum_{m,n\geq 0} \binom{k+n}{m}\binom{b+m}{n} x^m y^n. \qquad \text{(Pfaff-Saalschütz-GS)}$$

Indeed, the original Pfaff-Saalschütz identity (which from our point of view is simpler than this equivalent form, and that was proved easily in [WZ1]) is obtained by extracting the coefficient of $x^m y^n$ from both sides. The computer proof runs as follows.

Theorem 6.2.2. *Let*

$$F(x,y,k,b) = \frac{m!n!(k+n-m)!(b+m-n)!(1+x)^k(1+y)^b}{(1-xy)^{k+b+1}x^{m+1}y^{n+1}(k+n)!(b+m)!}$$

and let $a(k,b)$ be its double contour integral w.r.t. x and y. Then the discrete function $a(k,b)$ satisfies the recurrence $\Delta_k a(k,b) = 0$.

Proof. It is routinely verifiable that

$$\Delta_k F(x,y,k,b) = D_x\Big(\frac{x(1+x)}{(1-xy)(k+n+1)}F(x,y,k,b)\Big) + D_y\Big(\frac{xy(1+y)}{(k+n+1)(xy-1)}F(x,y,k,b)\Big),$$

and the result follows by double contour-integrating w.r.t. x and y. By symmetry of k and b, also $\Delta_b a(k,b) = 0$, and since trivially $a(0,0) = 1$, it follows that $a(k,b) \equiv 1$. $\square$

6.2.3 The 3-Dimensional Mehta-Dyson integral

The Mehta-Dyson integral (Mehta-Dyson) (section 1.3) follows from (Selberg) by a simple limiting argument, as was first observed by Bombieri and Selberg (see [Ma]). Askey [As1] raised the question of a direct proof, which was answered by G. Anderson [Ande] who proved it on the lines of his proof [Ande1] of the finite field analog of Selberg's integral (conjectured by R. Evans [Ev]). Using the method of the present paper, we [WZ3] found yet another proof, which we feel is the shortest and simplest to date. Here we present only the proof of the case $n = 3$, which led to the general proof in [WZ3].

Theorem 6.2.3. *Let $F(x,y,z,c)$ be*

$$e^{-(x^2+y^2+z^2)/2}[(x-y)(x-z)(y-z)]^{2c},$$

and let $I(c)$ be its triple integral w.r.t. x, y, and z. Let C be the forward shift operator in c. The sequence $I(c)$ satisfies the recurrence

$$(6(3c+2)(2c+1)(3c+1) - C)I(c) = 0.$$

Proof. Let

$$\begin{aligned} P(x,y,z) &:= (-3z - 3y - 2z^3 - 2y^3 + 2y^2z^3 + 2y^3z^2 + 6c^2y - 3cz + 6c^2z - 6cz^3 - 3cy - 6cy^3) \\ &\quad + (4 + 24c^2 + 12c - 2y^2z^2 + 2y^2 - 6yz + 2z^2 - 6cyz)x + (-y^3 - z^3)x^2 + (-2yz + z^2 + y^2)x^3. \end{aligned}$$

It is then routinely verifiable that

$$(6(3c+2)(2c+1)(3c+1)-C)F(x,y,z,c)=$$
$$D_x(P(x,y,z)F(x,y,z,c))+D_y(P(y,x,z)F(x,y,z,c))+D_z(P(z,x,y)F(x,y,z,c))$$

and the result follows by triple integration w.r.t. x, y, and z. □

Another example of the use of the present algorithm to evaluate a double integral is given in [Ek3].

6.3 Sums/integrals.

The bilinear generating functions for the Hermite, Laguerre, and Jacobi polynomials, associated with the names of Mehler, Hille-Hardy, and Bailey respectively, have received considerable attention in recent years. This was due to Foata's (see [Fo]) beautiful combinatorial approach that lead to insightful and beautiful combinatorial proofs of the Mehler [Fo1] and Hille-Hardy [FS] identities. These beautiful proofs were naturally extended by Foata and Garsia [FoG] and Foata and Strehl [FS] respectively, leading to multivariate generalizations. As far as we know, there is still no Foata-style proof of the Bailey formula, but a short and elegant analytic proof was given by Stanton [Sta].

The present method produced new proofs to the identities, that, we believe, give a different kind of insight, which may suggest other generalizations. The Mehler formula involves only one integration, and was already given a computerized proof in [AZ].

6.3.1 The Hille-Hardy bilinear formula for Laguerre polynomials

The Hille-Hardy formula is considered by Askey to be "very important" ([As2], p. 34). Although we can prove it in the expanded form given in section 1.1 that equates a triple sum with a single sum, we prefer not to go all the way but rather "extract" the coefficient of u^n from the right. We will show the computer-proof of the equivalent version

$$\frac{n!}{(\alpha+1)_n}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)=\int_{|u|=\epsilon}u^{-n-1}(1-u)^{-\alpha-1}\exp\{-\frac{(x+y)u}{(1-u)}\}(\sum_m\frac{i}{m!(\alpha+1)_m}\left(\frac{xyu}{(1-u)^2}\right)^m)du. \tag{Hille-Hardy-semi-expanded}$$

To this end, it is enough to prove that the right side is annihilated by the well-known second order differential operator annihilating the Laguerre polynomials, both w.r.t. x and y. Of course, by symmetry, it suffices to do it only for x, but the computer doesn't mind doing it for both x and y. We still need to prove that the initial conditions match, but this is *really* trivial. The differential operator annihilating (Laguerre) is easily found by the single-sum [Z3] special case of our algorithm. The computer output was as follows.

Theorem 6.3.1. *Let*

$$F(u,m,x):=\frac{(1-u)^{-\alpha-1}}{u^{n+1}m!\Gamma(\alpha+1+m)}(\frac{xyu}{(1-u)^2})^m\exp(-\frac{(x+y)u}{(1-u)}),$$

and let $a(x)$ be its contour integral w.r.t. u and sum w.r.t. m. Let D_x be differentiation w.r.t. x. The function $a(x)$ satisfies the differential equation

$$(n+(\alpha+1-x)D_x+xD_x^2)a(x)=0.$$

Proof. It is routinely verifiable that

$$(n+(\alpha+1-x)D_x+xD_x^2)F(u,m,x)=D_u(-uF(u,m,x))+\Delta_m(-(m(\alpha+m)/x)F(u,m,x))$$

and the result follows by integrating w.r.t. u and summing w.r.t. m. $\square$

6.3.2 The Bailey bilinear formula for Jacobi polynomials

Bailey [Ba1] found the following bilinear generating function for Jacobi polynomials:

$$(1+t)^{\alpha+\beta+1}\sum_{n=0}^{\infty}\frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)t^n = \\ \sum_{k,m=0}^{\infty}\frac{(\alpha+\beta+1)_{2(m+k)}}{k!m!(\alpha+1)_k(\beta+1)_m}\left[\frac{t}{4(1+t)^2}\right]^{m+k}(1+x)^m(1+y)^m(1-x)^k(1-y)^k. \tag{Bailey}$$

Stanton [Sta] gave a short and elegant proof of (Bailey), but not as short, and not as elegant, as the proof below, found by our computer.

Theorem 6.3.2. *Let $F(t,k,m,x)$ be given by*

$$(1+t)^{-\alpha-\beta-1}\frac{(\alpha+\beta+1)_{2(m+k)}}{k!m!(\alpha+1)_k(\beta+1)_m}\left[\frac{t}{4(1+t)^2}\right]^{m+k}\frac{(1+x)^m(1+y)^m(1-x)^k(1-y)^k}{t^{n+1}},$$

and let $a(x)$ be its integral w.r.t. t, sum w.r.t. m, k. Let D_x be differentiation w.r.t. x. The function $a(x)$ satisfies the differential equation

$$((\alpha+\beta+1+n)n+(-\beta x-\alpha x-2x+\beta-\alpha)D_x+(1-x)(1+x)D_x^2)a(x)=0.$$

Proof. Let D_t be differentiation with respect to t and Δ_k and Δ_m be the forward shift operators in k and m respectively. It is routinely verifiable that

$$((\alpha+\beta+1+n)n+(-\beta x-\alpha x-2x+\beta-\alpha)D_x+(1-x)(1+x)D_x^2)F(t,k,m,x)= \\ D_t\left(\frac{(tm+tk-tn-\alpha-\beta-1-n-m-k)}{1+t}F(t,k,m,x)\right) \\ +\Delta_k\left(\frac{2k(k+\alpha)}{x-1}F(t,k,m,x)\right)+\Delta_m\left(\frac{-2m(m+\beta)}{1+x}F(t,k,m,x)\right),$$

and the result follows by contour integrating w.r.t. t, and summing w.r.t. k and m.

6.4 q-sums and integrals.

So far we have only the single-sum (and hence the single-integral) case implemented. In this case, in fact, one has even a faster algorithm [Z5], that is a q-analog of the algorithm of [Z3], and is based on a q-analog of Gosper's algorithm, that is also described in [Z5]. The q-Fundamental Theorem of the present paper, for the special case of a single summation variable, provides the theoretical *guarantee* that the q-fast algorithm of [Z5] always works.

We give three examples. The first is the celebrated q-Saalschütz identity (e.g [Z6]) as a typical example of a q-sum. The second example is a specialization of the q-Vandermonde identity, that while trivial, leads to a highly non-trivial *dual identity.* The third example is the q-Askey-Wilson identity [AsWi], as a representative example of a (contour-) integral, or equivalently, as a constant term identity.

6.4.1 A WZ proof of the q-Saalschütz identity

Theorem 6.4.1. *(Jackson) Let, as always,* $(q)_a := (1-q)(1-q^2)\dots(1-q^a)$, *then*

$$\sum_k \frac{q^{k^2-m^2}(q)_{n+b+c-k}}{(q)_{n-k}(q)_{b-k}(q)_{c-k}(q)_{k+m}(q)_{k-m}} = \frac{(q)_{n+b}(q)_{n+c}(q)_{b+c}}{(q)_{n+m}(q)_{b-m}(q)_{c+m}(q)_{n-m}(q)_{b+m}(q)_{c-m}}.$$

Proof. Let $F(n,k)$ be the summand on the left divided by the right side. We have to prove that

$$a(n) := \sum_k F(n,k) \equiv 1.$$

This is obviously true for $n=0$, so it suffices to prove that $a(n+1)-a(n) \equiv 0$. To this end, we construct

$$G(n,k) := -\frac{q^{n-k}(q^k-q^c)(q^k-q^b)}{(1-q^{n+c+1})(1-q^{n+b+1})}F(n,k),$$

with the motive that

$$\Delta_n F(n,k) = \Delta_k G(n,k-1), \text{ (check!)}$$

(dividing throughout by $F(n,k)$ results in a routinely verifiable identity), and the assertion follows upon summing w.r.t. k. $\square$

6.4.2 Another example of a q-WZ pair

We will now present a WZ proof of the following special case of the well-known, and almost trivial, q-Vandermonde-Chu identity.

$$\sum_k q^{k^2}\begin{bmatrix} n \\ k \end{bmatrix}^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}. \qquad (q\text{-Chu-equal})$$

Indeed, letting $F(n,k)$ be the summand on the left divided by the right side, we construct

$$G(n,k) := \frac{q^{n+1}(q^{2n+1}+q^n-2q^k)}{(1-q^{2n+1})(1+q^{n+1})}F(n,k),$$

with the motive that

$$F(n+1,k)-F(n,k) = G(n,k)-G(n,k-1),$$

and summing w.r.t. k implies that $\sum_k F(n,k)$ is independent of n, and hence equals its value at $n=0$, which is seen to be 1. $\square$

The advantage of the present proof is that, just as in the ordinary case [WZ2], we get two brand new identities as bonuses: the *dual* and the *companion*. First we present the *dual*, which is particularly attractive. As in [WZ2], we must first take the "shadow" of the whole or parts of $G(n,k)$ to make it of finite support in n for every k. The shadow of $(q)_n$ is $(-1)^n q^{(n+1)n/2}/(q)_{-n-1}$. Shadowing the part $(q)_n^2/(q)_{2n}$ in $G(n,k)$, changing n to $-n$, shifting, and finally switching n and k, to make k the summation variable, results in the following dual identity, which is the q-analog of example $3'$ (p. 155) of [WZ2].

$$\sum_{k=0}^{n}\begin{bmatrix} 2k \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{(q^{n-k}+q^n-2q^k)}{(1+q^k)} \equiv 0. \qquad (q\text{-Chu-equal-dual})$$

Unlike the ordinary case, in which letting $n \to \infty$ is meaningless or tautological, in the q-case, very often, we get an even nicer, this time *non-terminating*, identity. In the present case, letting $n \to \infty$ leads to

$$\sum_{k=0}^{\infty} \frac{q^k}{(q)_k^2} = (q)_\infty \left(1 + 2\sum_{k=1}^{\infty} \frac{q^k (q)_{2k-1}}{(q)_k^3 (q)_{k-1}}\right). \qquad (q\text{-Chu-dual-limiting})$$

This appears to be a new identity. When we presented it to George Andrews, he was able, of course, to prove it independently, but the proof was very long and used several esoteric results. At any rate, our unified method *discovered*, and simultaneously proved, a new identity. Undoubtedly, there are many other examples of *non-terminating* q-identities one could find, by specializing, dualizing, and then limiting known identities. Another example of the form "single sum = single sum", whose limiting case is the First Rogers-Ramanujan identity, is given in [EkT].

As regards the *companion* identity to (q-Chu-equal), it is found, following exactly the method of [WZ2], to be

$$\sum_{n \geq 0} \frac{q^{n+1}(q^{2n+1} + q^n - 2q^k)}{(1 - q^{2n+1})(1 + q^{n+1})} \frac{\begin{bmatrix} n \\ k \end{bmatrix}^2}{\begin{bmatrix} 2n \\ n \end{bmatrix}} = q^{-k^2}\left((q)_\infty \sum_{j=0}^{k} \frac{q^{j^2}}{(q)_j^2} - 1\right), \qquad (k \geq 0) \qquad (q\text{-Chu-companion})$$

and it appears to be new also. The case $k = 0$, which reads as

$$(q)_\infty = 1 + \sum_{n \geq 0} \frac{q^{n+1}(q^{2n+1} + q^n - 2)}{(1 - q^{2n+1})(1 + q^{n+1})} \frac{(q)_n^2}{(q)_{2n}}$$

is already of some interest.

6.4.3 The Askey-Wilson integral

The Askey-Wilson integral is equivalent to the following constant term identity (recall that CT_z is the constant term w.r.t. z.)

$$CT_z \frac{(z^2)_\infty (z^{-2})_\infty}{(az)_\infty (a/z)_\infty (bz)_\infty (b/z)_\infty (cz)_\infty (c/z)_\infty (dz)_\infty (d/z)_\infty} = \frac{2}{(q)_\infty} \frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}, \qquad (\text{Askey-Wilson})$$

where $|a|, |b|, |c|, |d| < 1$.

(Askey-Wilson) is needed for the proof of the orthogonality of the q-Askey-Wilson polynomials, which are the most general classical orthogonal polynomials known. The original evaluation in [AsWi] used Cauchy's theorem and a miraculous simplification of elliptic functions. Later Askey [As3] verified it by showing that both sides satisfy a certain functional equation. A beautiful combinatorial proof is given in [ISV]. The present computer-generated proof is most akin to that of [As3], but our functional equation is simpler (namely (WZ)), and, of course, is machine-made.

The proof goes as follows. Let $S(a, b, c, d)$ be the left side of (Askey-Wilson) divided by its right side. We would like to prove that it is identically 1. A direct computation, using the trivial $(q; q^2)_\infty(-q; q)_\infty = 1$ (of Euler's "odd-distinct" fame), shows that $S(1, -1, q^{1/2}, -q^{1/2}) = 1$. The result would follow by a density/analytic continuation argument, if we can show that $S(qa, b, c, d) \equiv S(a, b, c, d)$, and similarly for b, c, d, because it would imply it for all $\{a\}$ assuming the values $+q^n$, which for $q < 1$ converge to 0, just as in

Askey's [As3] proof. It remains to show that $(Q_a - 1)S(a,b,c,d) = 0$, where, as above, $Q_a f(a) := f(qa)$. Let $F(a,b,c,d,z)$ be the *constant termand* of $S(a,b,c,d)$, i.e. the left side of (Askey-Wilson) without the CT_z, divided by the right side. We construct

$$G(a,b,c,d,z) := -F(a,b,c,d,z)\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-ab)(1-ac)(1-ad)z},$$

with the motive that

$$(Q_a - 1)F(a,b,c,d,z) = (1 - Q_z^{-1})G(a,b,c,d,z) \quad \text{(check!)}$$

(dividing through by F results in a purely routine identity involving specific rational functions), and the results follow upon taking constant term w.r.t. z.

6.4.4 An ancient q-WZ proof

In [BFH] can be found a q-WZ proof that is attributed there to Cayley [Ca]. Our work shows that such a proof, possibly generalized as in our q-fundamental theorem, *always* exists, and one does not have to be a Cayley to find it.

6.5 From specifically many to arbitrarily many summations and integrations: the human factor.

As was pointed out in the introduction, at present it is possible to find electronically only proofs of identities with a *fixed* number of summation and/or integration signs. However, a human eye might detect a discernible pattern, that might be generalizable to *arbitrarily* many summation and integration signs, and then prove, humanly, the "one-line" WZ identity, whose proof might require several lines. We were able to find such proofs for Selberg's integral [Se] [An1] and Holman's [Ho] $U(n)$-Gauss summation. We are confident that in the future it will be possible to find similar proofs of other multivariate identities like those of Milne [Mi], Gustafson [Gu], and the q-Dyson identity [ZB].

Once one knows what to look for, it is conceivable that one may be able to dispense with the computer altogether, since general families of multivariate identities display elegant symmetries that can be used by humans to find "soft" proofs.

6.5.1 A WZ Proof of Selberg's Integral

Selberg's integral (Selberg), already mentioned in section 1.3,

$$\int_0^1 \cdots \int_0^1 \{\prod_{i=1}^n t_i^x (1-t_i)^y \prod_{1\le i<j\le n} (t_i - t_j)^{2z}\} dt_1 \dots dt_n = \prod_{j=1}^n \frac{(x+(j-1)z)!(y+(j-1)z)!(jz)!}{(x+y+(n+j-2)z+1)!z!}, \quad \text{(Selberg)}$$

is proved as follows. The easy part is the same as in Selberg's original proof ([An1]): when $x = y = 0$, the integral can be transformed by symmetry, to n times the integral over $0 \le t_1, t_2, \dots, t_{n-1} < t_n < 1$, and the change of variables $t_i = s_i t_n$, $(i = 1, \dots, n-1)$, transforms the integral to a special case of the $(n-1)$-variate case. Let $S(x,y,z)$ be the left side of (Selberg) divided by its right side. We would like to show that $S(x,y,z) \equiv 1$. By induction it is enough to show that $S(x+1,y,z) - S(x,y,z) \equiv 0$ and $S(x,y+1,z) - S(x,y,z) \equiv 0$. By symmetry it is enough to show the former (or the latter). Let $F(x,y,z;t_1,\dots,t_n)$ be the integrand of (Selberg) divided by its right side. Let $e_j(v_1,\dots,v_m)$ be the elementary symmetric functions in $v_1,\dots,v_m$:

$$\prod_{i=1}^{m}(x+v_i)=\sum_{j=0}^{m}e_j(v_1,\dots,v_m)x^{m-j}.$$

We construct

$$R(u;v_1,\dots,v_{n-1}):=u(1-u)\sum_{j=0}^{n-1}\frac{-(j!)(n-j-1)!}{n!(x+1+(n-1)z)}\Big(\prod_{r=1}^{j}\frac{x+y+(2n-r-1)z+2}{x+1+z(n-1-r)}\Big)e_j(v_1,\dots,v_{n-1}),$$

with the motive that

$$\Delta_x F(x,y,z;t_1,\dots,t_n)=\sum_{i=1}^{n}D_{t_i}\big[R(t_i;t_1,\dots,\hat{t_i},\dots,t_n)F(x,y,z;t_1,\dots,t_n)\big], \qquad \text{(WZ-Selberg)}$$

whose truth would imply $S(x+1,y,z)-S(x,y,z)\equiv 0$, by integrating w.r.t. $t_1,\dots,t_n$ over $[0,1]^n$. The proof of (WZ-Selberg) is, from this point on, no longer purely routine, although it is for every *specific* n. Although not machine-provable (at present), it is nevertheless an easy exercise that uses the following simple identities.

$$\sum_{i=1}^{n}e_j(t_1,\dots,\hat{t_i},\dots,t_n)=(n-j)e_j(t_1,\dots,t_n), \qquad \text{(i)}$$

$$\sum_{i=1}^{n}t_ie_j(t_1,\dots,\hat{t_i},\dots,t_n)=(j+1)e_{j+1}(t_1,\dots,t_n), \qquad \text{(ii)}$$

$$\sum_{i=1}^{n}\Big(\sum_{\substack{1\le j\le n\\ j\ne i}}\frac{1}{t_i-t_j}\Big)t_i(1-t_i)e_r(t_1,\dots,\hat{t_i},\dots,t_n)=\binom{n-r}{2}e_r(t_1,\dots,t_n)-\frac{(r+1)(2n-r-2)}{2}e_{r+1}(t_1,\dots,t_n). \qquad \text{(iii)}$$

The proofs of (i)-(iii) and their use to prove (WZ-Selberg) are left as pleasant exercises to the reader. □

The present proof is somewhat reminiscent of Aomoto's [Ao] beautiful but *ad hoc* proof of Selberg's integral.

6.5.2 A WZ Proof of Holman's U(n)-Gauss summation

One of the first general multivariate hypergeometric identities, that inspired the impressive work of Milne (e.g. [Mi]) and Gustafson (e.g. [Gu]) was Holman's $U(n)$ summation formula, that generalizes from one to many summation variables Gauss' classical evaluation of ${}_2F_1(1)$. It can be stated as follows.

$$\sum_{y_1=0}^{a_1}\cdots\sum_{y_n=0}^{a_n}\frac{\prod_{1\le i<j\le n}(y_i-y_j+B_i-B_j)\prod_{i=1}^{n}\prod_{j=1}^{n}(-a_i+B_j-B_i)_{y_j}\prod_{j=1}^{n}(a+B_j)_{y_j}}{\prod_{i=1}^{n}\prod_{j=1}^{n}(1+B_j-B_i)_{y_j}\prod_{j=1}^{n}(b+B_j)_{y_j}}$$
$$=\frac{(b-a)_{a_1+\cdots+a_n}\prod_{1\le i<j\le n}(B_i-B_j)}{(b)_{a_1}(b-B_2)_{a_2}\dots(b-B_n)_{a_n}}. \qquad \text{(Holman)}$$

Here $a_1, \ldots, a_n$ are non-negative integers, while $B_1 = 0, B_2, \ldots, B_n$, b and a are commuting indeterminates. By dividing by the right side, (Holman) can be written as

$$\sum_{y_1,\ldots,y_n} F_n(a_1, y_1, \ldots, y_n) = 1,$$

where the dependence on $a_2, \ldots, a_n$, is suppressed, as is the dependence on the other parameters B_j, a and b. It is immediate to see that when $a_1 = 0$, the identity reduces to the $n-1$ case. Calling the left side $H(a_1)$ it would follow that $H(a_1) \equiv 1$, once we can prove that $H(a_1+1) - H(a_1) \equiv 0$. This, in turn, would be an immediate consequence of the existence of rational functions R_i such that (recall that $\Delta_v f(v) := f(v+1) - f(v)$),

$$\Delta_{a_1} F_n = \sum_{i=1}^{n} \Delta_{y_i}(R_i F_n), \tag{WZ-Holman}$$

which would imply it by summing w.r.t. $y_1, \ldots, y_n$.

The fundamental theorem guarantees that for each specific n, such rational functions exist, (possibly with the Δ_{a_1} on the left replaced by a certain left multiple of it), and that a computer can find them. Our computer found them for the cases $n = 2, 3$. To our pleasant surprise, these factorized nicely, and we were able to conjecture, and then prove, the general expression:

$$R_i = \frac{(y_i + b + B_i - 1)\prod_{j=1}^{n}(y_i + B_i - B_j)}{(b - a + a_1 + \cdots + a_n)(y_i - a_1 + B_i - 1)\prod_{\substack{1 \le j \le n \\ j \neq i}}(B_i + y_i - B_j - y_j)}.$$

The verification of (WZ-Holman) for *every specific* n, after dividing throughout by F_n, is a purely routine identity involving sums of rational functions. However, the statement *for a general* n requires a human proof. This proof however, is an easy exercise that uses the Lagrange interpolation formula, and goes as follows.

By dividing (WZ-Holman) by F_n, it emerges that it is equivalent to the following identity, where we have set $z_i := y_i + B_i$.

$$\frac{(b + a_1)\prod_{j=1}^{n}(-a_1 + B_j - 1)}{(b - a + a_1 + \cdots + a_n)\prod_{j=1}^{n}(z_j - a_1 - 1)} - 1 =$$
$$\sum_{i=1}^{n} \frac{1}{(z_i - z_1)\ldots(z_i - z_{i-1})(z_i - z_{i+1})\ldots(z_i - z_n)} \times$$
$$\left[\frac{(a + z_i)}{(b - a + a_1 + \cdots + a_n)}\prod_{j=2}^{n}(-a_j - B_j + z_i) - \frac{(z_i + b - 1)}{(z_i - a_1 - 1)(b - a + a_1 + \cdots + a_n)}\prod_{j=1}^{n}(z_i - B_j)\right], \tag{Rational}$$

which is an easy consequence of the Lagrange interpolation formula. It is proved as follows.

(i) In (Rational), replace $(z_i + b - 1)/(z_i - a_1 - 1)$ by $1 + (b + a_1)/(z_i - a_1 - 1)$.

(ii) Recall the formula

$$\mathrm{Coeff}_{z^{n-1}} P(z) = \sum_{i=1}^{n} \frac{P(w_i)}{(w_i - w_1)\ldots(w_i - w_{i-1})(w_i - w_{i+1})\ldots(w_i - w_n)} \tag{Lagrange$'$}$$

obtained by taking the coefficient of z^{n-1} in the Lagrange Interpolation Formula:

$$P(z) = \sum_{i=1}^{n} \frac{P(w_i)(z - w_1)\ldots(z - w_{i-1})(z - w_{i+1})\ldots(z - w_n)}{(w_i - w_1)\ldots(w_i - w_{i-1})(w_i - w_{i+1})\ldots(w_i - w_n)}, \tag{Lagrange}$$

which is valid for any polynomial $P(z)$ of degree $n-1$ and n points $w_1, \ldots, w_n$.

(iii) Apply (Lagrange$'$) to the degree-n polynomial $P(z) := \prod_{j=1}^{n}(z - B_j)$ at the $n+1$ points

$$\{z_1, \ldots, z_n, a_1 + 1\}.$$

(iv) Apply (Lagrange$'$) to the degree-$(n-1)$ polynomial

$$P(z) := (a + z) \prod_{j=2}^{n}(z - a_j - B_j) - \prod_{j=1}^{n}(z - B_j)$$

at the n points $\{z_1, \ldots, z_n\}$ and add to (iii) to obtain (i). $\square$

Acknowledgments. We thank our colleagues with whom we have discussed this work and who have offered valuable suggestions and insights. They include Gert Almkvist, Richard Askey, Joseph Bernstein, Pierre Cartier, Philippe Flajolet, Dominique Foata, Ira Gessel, Roger Howe, Donald Knuth, Peter Paule, Marko Petkovšek and Volker Strehl. We also thank Shalosh B. Ekhad for its diligent computations. Special thanks are due to Dr. James C. T. Pool, Head of the Drexel University Department of Mathematics and Computer Science, for his generous permission to use his department's computing facilities. Most of the examples of section 6 were done on Drexel's computer <`king.mcs.drexel.edu`>.

References

[AZ] G. Almkvist and D. Zeilberger, *The method of differentiating under the integral sign*, (The Maple program that implements the algorithm appeared in ACM SIGSAM Bulletin **25**, No. 3 (July 1991), 14-17), J. Symbolic Computation **10** (1990), 571–591.

[Ande] G. Anderson, Letter to R. Askey.

[Ande1] ———, *The evaluation of Selberg sums*, C. R. Acad. Sci. (Ser. I: Math.) **311** (1990), 469–472.

[An] G. Andrews, *The Theory of Partitions, Encyclopedia of Mathematics and Its Applications (G. C. Rota, Editor)*, vol. 2, Addison Wesley, Reading, 1976.

[An1] ———, *q-Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra, CBMS series*, vol. 66, Amer. Math. Soc., Providence,, 1986..

[Ao] K. Aomoto, *Jacobi polynomials associated with Selberg's integrals*, SIAM J. Math. Anal. **18** (1987), 545–549.

[As] R. A. Askey, *Orthogonal Polynomials and Special Functions*, Regional Conference Series in Applied Mathematics, vol. 21, SIAM, 1975.

[As1] ———, *Lost and found mathematics*, MAA-AAAS invited address, Annual meeting, Columbus, OH, Aug. 1990.

[As2] ———, *Ramanujan and important formulas*, Srinivasa Ramanujan, a tribute (N. R. Nagarajan and T. Soundararajan, eds.), Macmillan, India, Madras, 1988.

[As3] ———, *An elementary evaluation of a beta type integral*, Indian J. Pure Appl. Math. **14** (1983), 892–895.

[AsWi] R. Askey and J. A. Wilson, *Some basic hypergeometric polynomials that generalize the Jacobi polynomials*, Memoirs Amer. Math. Soc. **319** (1985).

[Ba] W. N. Bailey, *Generalized Hypergeometric Series, Cambridge Math. Tract No. 32*, (Reprinted: Hafner, New York, 1964), Cambr. Univ. Press, London and New York, 1935.

[Ba1] ———, *The generating function for Jacobi polynomials*, J. London Math. Soc. **13** (1938), 243–246.

[BFH] D. Benson, W. Feit, and R. Howe, *Finite linear groups, the Commodore 64, Euler, and Sylvester*, Amer. Math. Monthly **93** (1986), 717–720.

[Be] I. (J.) N. Bernstein, *Modules over rings of differential operators, a study of the fundamental solutions of equations with constant coefficients*, Funct. Anal. and Appl. **5** (1971), 1–16.

[Bj] J. E. Björk, *Rings of Differential Operators*, North–Holland, Amsterdam, 1979.

[Bo] A. Borel et. al., *Algebraic D-Modules*, Academic Press, Boston, 1987.

[Ca] A. Cayley, *Collected Papers*, vol. 12, pp. 217–219.

[Co] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[Ca1] L. Carlitz,, *Summations of products of binomial coefficients*, Amer. Math. Monthly **75** (1968), 906–908.

[Ca2] ———, *A binomial identity arising from a sorting problem*, SIAM Review **6** (1964), 20–30.

[deB] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[Eg] G. P. Egorychev, *Integral Representation and the Computation of Combinatorial Sums*, vol. 59, American Mathematical Society Translations, Providence, 1984.

[Eh] F. Ehlers, *The Weyl algebra, in [Bo]*, 173–205.

[Ek1] S. B. Ekhad, *Short proofs of two hypergeometric summation formulas of Karlsson*, Proc. Amer. Math. Soc. **107** (1989), 1143–1144.

[Ek2] ———, *A very short proof of Dixon's theorem*, J. Comb. Theory, Ser. A **54** (1990), 141–142.

[Ek3] ———, *A one-line proof of the Habsieger–Zeilberger G_2 constant term identity*, J. Comput. Appl. Math. **34** (1991), 133–134.

[Ek4] ———, *Short proof of a strange combinatorial identity conjectured by Gosper*, Discrete Math. **90** (1991), 319–320.

[Ek5] ———, *A Short, elementary, and easy, WZ proof of the Askey–Gasper inequality that was used by de Branges in his proof of the Bieberbach conjecture*, J. Theor. Comp. Sci (to appear).

[ET] S. B. Ekhad and S. Tre, *A purely verification proof of the first Rogers–Ramanujan identity*, J. Comb. Theory Ser. A **54** (1990), 309–311.

[Er] A. Erdélyi et al., *The Higher Transcendental Functions*, The Bateman Manuscript Project, 3 vols., McGraw-Hill, New York, 1953.

[Ev] R. Evans, *Identities for products of Gauss sums over finite fields*, Enseign. Math. **27** (1981), 197–209.

[EZ] S. B. Ekhad and D. Zeilberger, *A 21st century proof of Dougall's hypergeometric sum identity*, J. Math. Anal. Appl. **147** (1990), 610–611.

[Fa] M. C. Fasenmyer, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. **53** (1947), 806–812.

[Fo] D. Foata, *Combinatoire des identites sur les polynomes orthogonaux*, Proc. ICM 83, Varsovie (1983), 1541–1553.

[Fo1] ———, *A combinatorial proof of the Mehler formula*, J. Comb. Theory, ser. A **24** (1978), 250-259.

[FG] D. Foata and A. Garsia, *A combinatorial approach to the Mehler formulas for Hermite polynomials*, Relations between combinatorics and other branches of mathematics, Columbus, 1978, Amer. Math. Soc., Providence, 1979, pp. 163–179.

[FS] D. Foata and V. Strehl, *Une extension multilinéaire de la formule d'Erdélyi pour les produits de fonctions hypergéometriques confluentes*, C.R. Acad. Sc. Paris **293** (1981), 517–520.

[GaGo] [redacted]. Garvan and G. Gonnet, *Macdonald's constant term conjectures for the exceptional root systems*, Bulletin (new ser.) of the Amer. Math. Soc. **24** (1991), 343–347.

[GaRa] G. Gasper and M. Rahman, *Basic Hypergeometric Series, Encyclopedia of Mathematics and Its Applications*, Cambridge Univ. Press, Cambridge, 1990.

[GS] I. Gessel and D. Stanton, *Short proofs of Saalschütz' and Dixon's theorems*, J. Combin. Theory A **38** (1985), 87–90.

[Goo] I. J. Good, *Short proof of a conjecture by Dyson*, J. Math. Phys. **11** (1970), 1884.

[Gos] Gosper, R. W., Jr., *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA **75** (1978), 40–42.

[GKP] Graham, R. L., Knuth, D. E., and Patashnik. O., *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1989.

[Gu] R. A. Gustafson, *A generalization of Selberg's beta integral*, Bull. (new series) Amer. Math. Soc. **22** (1990), 97–105.

[GuMi] R. A. Gustafson and S. C. Milne, *Schur functions, Good's identity, and hypergeometric series well poised in* $SU(n)$, Adv. in Math. **57** (1985), 209–225.

[Ho] W. J. Holman III, *Summation theorems for hypergeometric series in* $U(n)$, SIAM J. Math. Anal. **11** (1980), 523–532.

[ISV] M. Ismail, D. Stanton, and G. Viennot, *The combinatorics of* q*-Hermite polynomials and the Askey-Wilson integral*, Europ. J. Combin. **8** (1987), 379–392.

[K] Koornwinder, T. H., *Handling hypergeometric series in Maple*, Orthogonal Polynomials and their Applications (C. Brezinski, L. Gori and A. Ronveaux, eds.), IMACS Annals on computing and applied mathematics, Baltzer, 1991, pp. 73–80.

[L] Leonard Lipshitz, *The diagonal of a D-finite power series is D-finite*, J. Algebra **13** (1988), 373–378.

[Ma] I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 988–1007.

[Mi] S. C. Milne, *A* q*-analog of the Gauss summation theorem for hypergeometric series in* $U(n)$, Adv. Math. **72** (1988), 59–131.

[O] E. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math. **98** (1989), 1–18.

[R] E. D. Rainville, *Special Functions*, (Reprinted: Chelsea, Bronx, NY, 1971), Macmillan, New York, 1960.

[Ri] D. St. P. Richards, *Analogs and extensions of Selberg's integral*, q-Series and Partitions, IMA volumes in Mathematics and its Applications (D. Stanton, ed.), vol. 18, Springer, New York, 1989.

[Se] A. Selberg, *Bemerkninger om et multipelt integral*, Norsk Math. Tidsskr. **26** (1944), 71–78.

[Sta] D. Stanton, *A short proof of a generating function for Jacobi polynomials*, Proc. Amer. Math. Soc. **80** (1980), 398–400.

[St] V. Strehl, *Zykel-Enumeration bei Local-Strukturien Funktionen*, Universität Erlangen–Nürnberg, Germany, 1990.

[SzVa] Szondy and Varga, *A conjectured double sum*, preprint.

[V] P. Verbaeten, *The automatic construction of pure recurrence relations*, Proc. EUROSAM '74, ACM–SIGSAM Bulletin **8** (1974), 96–98.

[WZ1] H. S. Wilf and D. Zeilberger, *Towards computerized proofs of identities*, Bull. Amer. Math. Soc. **23** (1990), 77–83.

[WZ2] ———, *Rational functions certify combinatorial identities*, J. Amer. Math. Soc. **3** (1990), 147–158.

[WZ3] ———, *A WZ evaluation of the Mehta-Dyson integral*, in preparation.

[Z0] D. Zeilberger, *Sister Celine's technique and its generalizations*, J. Math. Anal. Appl. **85** (1982), 114–145.

[Z1] ———, *A holonomic systems approach to special functions identities*, J. of Computational and Applied Math. **32** (1990), 321–368.

[Z2] ———, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math **80** (1990), 207–211.

[Z3] ———, *The method of creative telescoping*, J. Symbolic Computation **11** (1991), 195–204. (The Maple program that implements the algorithm appeared in ACM SIGSAM Bulletin 1991, vol. 25, pp. 4–13.

[Z4] ———, *Closed form (pun intended!)*, Special volume in memory of Emil Grosswald (M. Knopp, ed.), Contemporary Mathematics, Amer. Math. Soc. (to appear).

[Z5] ———, *The method of creative telescoping for* q*-series*, in preparation.

[Z6] ———, *A* q*-Foata proof of the* q*-Saalschütz identity*, European J. Combinatorics **8** (1987), 461–463.

[ZB] D. Zeilberger and D. Bressoud, *A proof of Andrews'* q*-Dyson conjecture*, Discrete Math **54** (1985), 201–224.