On the coefficients in the Hardy-Ramanujan-Rademacher formula for $p(n)$

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The number of partitions of n is given by the asymptotic and convergent series

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(x-\frac{1}{24})}\right)}{\sqrt{(x-\frac{1}{24})}} \right]_{x=n},$$

in which

$$A_k(n) = \sum_{\substack{(h,k)=1 \\ 1 \le h < k}} \omega_{h,k} e^{-2\pi i n h/k}$$

and $\omega_{k,h} = \exp(\pi i s(h,k))$ where

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \lfloor \frac{\mu}{k} \rfloor - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \lfloor \frac{h\mu}{k} \rfloor - \frac{1}{2} \right).$$

This formula represents the culmination of fundamental researches of Hardy and Ramanujan [HR] and of Rademacher [R]. The formula can actually be truncated after $C\sqrt{n}$ terms, and the exact value of $p(n)$ will be the nearest integer to that partial sum.

In the summer of 1992 a conference to honor the centenary of Rademacher's birth was held at Pennsylvania State University, and there one of us (H.W.) conjectured the truth of the following theorem. Shortly thereafter the theorem was proved by the first author [A]. In this paper we give a shorter proof of the result, using a different approach to the multiplicative nature of the $A_k(n)$'s.

Theorem. *For $k \ge 1$, if we define the polynomial*

$$f_k(x) = \prod_{n=1}^{k} (x - A_k(n)) \tag{1}$$

then the coefficients of all powers of x are always in $Z(\sqrt{k})$. They are in fact all (rational) integers if and only if k is even or a square.

Thus for example,

$$f_9(x) = -243 + 729x - 810x^3 + 243x^5 - 27x^7 + x^9,$$

while

$$f_{11}(x) = -1331x^5 - 242\sqrt{11}x^6 + 726x^7 + 11\sqrt{11}x^8 - 55x^9 + x^{11}.$$

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The proof of the theorem is simplified greatly by dealing with the power sums of the zeros of $f_k(x)$ rather than directly with their elementary symmetric functions. First, Lehmer [L] and Whiteman [W] have shown that in the case where k is a prime power each $A_k(n)$ can be expressed in the form $\alpha \cos r\pi$ or $\alpha \sin r\pi$, where α is an integer multiple of $\sqrt{k}$ and r is rational.

From these formulas we can quite explicitly compute the power sums

$$S_r = S_r(f_k) = \sum_{n=1}^{k} A_k(n)^r \qquad (r = 1, 2, \ldots, k),$$

which of course completely determine the coefficients of $f_k(x)$. It will then be easy to see that the theorem is correct if k is a prime power.

Next, since it is also known, by the work of [L] and [W], that the $A_k(n)$'s are in a certain sense multiplicative functions of n for fixed k, it will follow that if $(k', k'') = 1$ then the rth power sum of $f_{k'k''}(x)$ is simply the product $S_r(f_{k'})S_r(f_{k''})$, which will prove the result in general.

These formulas will be very explicit, so that in the end we obtain explicit formulas for the power sums of the $A_k(n)$'s for arbitrary k (see equations (8) and (9) below), and so by Newton's identities we have a quick way of computing the coefficients of the polynomials (1).

The results of Lehmer and Whiteman

We write $(\frac{a}{b})$ for the Legendre symbol, and congruences whose moduli are otherwise unspecified are to the modulus k. The following results were found for prime power values of k.

First, if $k = p^a$, where p is a prime ≥ 5, then put $v = 1 - 24n$, and we have

$$A_k(n) = \begin{cases} 0 & \text{if } v \text{ is a quadratic nonresidue of } k \text{ and prime to } k; \\ 2(\frac{3}{k})\sqrt{k}\cos(\frac{4\pi m}{k}) & \text{if } v \equiv (24m)^2 \text{ and is prime to } k; \\ (\frac{3}{k})\sqrt{k} & \text{if } v \equiv 0 \text{ and } a = 1; \\ 0 & \text{if } v \equiv 0 \text{ and } a > 1. \end{cases} \tag{2}$$

If $k = 3^a$ then

$$A_k(n) = 2(-1)^{a+1}(\frac{m}{3})\sqrt{\frac{k}{3}}\sin(\frac{4\pi m}{3k}), \tag{3}$$

where m is an integer s.t. $(8m)^2 \equiv 1 - 24n \pmod{3k}$. If $k = 2^a$, $a \geq 0$, then

$$A_k(n) = (-1)^a(\frac{-1}{m})\sqrt{k}\sin(\frac{\pi m}{2k}),$$

where m is an integer such that $(3m)^2 \equiv 1 - 24n \pmod{8k}$.

Next suppose k is not a prime power, and let $k = k_1 k_2$ with $(k_1, k_2) = 1$. Then we have the following multiplicative properties of the $A_k(n)$'s.

If k is either odd or a multiple of 8, then $A_k(n) = A_{k_1}(n_1)A_{k_2}(n_2)$, where n_1 and n_2 are determined by the congruences

$$
\begin{aligned}
k_2^2 d_2 e n_1 &\equiv d_2 e n + (k_2^2 - 1)/d_1 \pmod{k_1} \\
k_1^2 d_1 e n_2 &\equiv d_1 e n + (k_1^2 - 1)/d_2 \pmod{k_2},
\end{aligned}
$$

and where d_1, d_2, e are defined by

$$d_1 = (24, k_1), \quad d_2 = (24, k_2), \quad 24 = d_1 d_2 e.$$

If $k = 4k_1$ with k_1 odd then $A_k(n) = -A_{k_1}(n_1)A_4(n_2)$, where n_1, n_2 are determined by

$$128 n_1 \equiv 8n + 5 \pmod{k_1} \quad \text{and} \quad k_1^2 n_2 \equiv n - 2 - (k_1^2 - 1)/8 \pmod{4}.$$

Finally, if $k = 2k_1$, k_1 odd, then $A_k(n) = A_{k_1}(n_1)A_2(n_2)$, where n_1, n_2 are determined by

$$32 n_1 \equiv 8n + 1 \pmod{k_1} \qquad \text{and} \qquad n_2 \equiv n - (k_1^2 - 1)/8 \pmod{2}.$$

From the foregoing one can show (see Lemma 4 of [A]) that if $k = k_1 k_2$ with $(k_1, k_2) = 1$ then *the set of all* $\{A_k(n)\}_{n=1}^{k}$ *is identical with the set of all products* $A_{k_1}(n_1)A_{k_2}(n_2)$ *as* n_1, n_2 *run independently over* $[1, k_1]$ *and* $[1, k_2]$.

The power sums

As a consequence of the last remark in the previous section we have

Theorem A. *Let* $S_r(k)$ *denote the sum of the* r*th powers of the zeros of the polynomial* $f_k(x)$ *of (1). If* $(k_1, k_2) = 1$ *then for all* r *we have* $S_r(k) = S_r(k_1)S_r(k_2)$. ∎

In this section we will compute quite explicit formulas for the power sums of the zeros of $f_k(x)$, from which the conclusions of the theorem will be obvious. The computations require only the following formulas, in which r is a nonnegative integer:

$$\sum_{n=0}^{k-1} \sin^{2r}\left(\frac{2\pi m n}{k}\right) = \frac{k}{2^{2r}} \sum_t (-1)^{\frac{kt}{2m}} \binom{2r}{r + \frac{kt}{2m}} \tag{4}$$

$$\sum_{n=0}^{k-1} \sin^{2r+1}\left(\frac{2\pi m n}{k}\right) = 0 \tag{5}$$

$$\sum_{n=0}^{k-1} \cos^{r}\left(\frac{2\pi m n}{k}\right) = 2^{-r} k \sum_t \binom{r}{\frac{r}{2} + \frac{kt}{2m}} \tag{6}$$

in which the binomial coefficient is understood to vanish if its lower argument is not a nonnegative integer or exceeds its upper argument. The sums on the right are over all integers

t, but of course they have compact support. These formulas can easily be checked by replacing the trigonometric functions by complex exponentials, using the binomial theorem, and interchanging the order of summation.

Consider the case where $k = p^a$, $p \geq 5$, $a \geq 2$, and put $S_1(r,k) = \sum_{n=0}^{k-1} A_n(k)^r$. By (2), that sum is one half of the sum

$$\sum_{\substack{1 \leq m \leq k \\ p \nmid m}} \{2(\frac{3}{k})\sqrt{k}\cos(\frac{4\pi m}{k})\}^r.$$

Using (6) we find easily that

$$S_1(r,k) = \frac{1}{2}(\frac{3}{k})^r k^{1+r/2}\left(\sum_t \binom{r}{\frac{r}{2}+\frac{kt}{4}} - \frac{1}{p}\sum_t \binom{r}{\frac{r}{2}+\frac{kt}{4p}}\right) \quad (\text{if } k = p^a, p \geq 5, a \geq 2). \tag{7}$$

The remaining cases follow similarly, and all of the prime power cases are summarized below:

$$S(r,k) = \begin{cases} S_1(r,k) \stackrel{\text{def}}{=} \frac{1}{2}(\frac{3}{k})^r k^{1+r/2}\left(\sum_t \binom{r}{\frac{r}{2}+\frac{kt}{4}} - \frac{1}{p}\sum_t \binom{r}{\frac{r}{2}+\frac{kt}{4p}}\right) & \text{if } k = p^a, p \geq 5, a \geq 2; \\ S_2(r,k) \stackrel{\text{def}}{=} k^{(r/2+1)} 2^{-r} \sum_t (-1)^t \binom{r}{\frac{r}{2}+tk} & \text{if } k = 2^a, a \geq 0; \\ S_3(r,k) \stackrel{\text{def}}{=} 3^{-\frac{r}{2}} i^{-r} k^{1+\frac{r}{2}} \sum_t \binom{r}{\frac{r}{2}+\frac{kt}{2}} e^{\frac{4\pi i t}{3}} (-1)^{\frac{r}{2}+\frac{kt}{2}+t} & \text{if } k = 3^a, a \geq 0; \\ S_4(r,k) \stackrel{\text{def}}{=} (\frac{3}{k})^r k^{r/2}\left(1 - 2^{r-1} + \frac{k}{2}\sum_t \binom{r}{\frac{r}{2}+\frac{tk}{4}}\right) & \text{if } k = p \geq 5. \end{cases} \tag{8}$$

Proof of the theorem

Let k be a fixed positive integer, and write $k = 2^a 3^b \sigma\tau$, where $\sigma = p_1 p_2 \cdots p_u$ is squarefree, and $(\sigma, 6) = 1$, and $\tau = q_1^{c_1} \cdots q_v^{c_v}$ has all $c_i > 1$ and is also prime to 6. Then by Theorem A above, the numbers $\{A_k(n)\}_{n=1}^k$ consist of all possible products

$$A_{2^a}(n_1) A_{3^b}(n_2) A_{p_1}(n_3) A_{p_2}(n_4) \cdots$$

as $n_1, n_2, \ldots$ run independently over $[1, 2^a], [1, 3^b], \ldots$. Hence, for the sum of the rth powers of the $A_k(n)$'s we have

$$S(r,k) = S_2(r, 2^a) S_3(r, 3^b) \{\prod_{p \backslash \sigma} S_4(r,p)\} \{\prod_{q \backslash \tau} S_1(r, q^c)\}, \tag{9}$$

where the S_i are given by (8) above.

Each of the formulas (8) yields a rational multiple of $k^{r/2}$. Thus,*all of the power sums for even values of* r *are rational numbers.*

Suppose k is a square. Then clearly all of the power sums are rational.

Suppose k is even. Then the factor $S_2(r, 2^a)$ actually appears with $a > 0$, in (8). But this factor is nonzero only if r is even. Hence $S(r, k)$ itself vanishes unless r is even. But if r is even then $k^{r/2}$ is an integer, so all of the powers sums are again rational.

Thus, if k is even or a square, all of the power sums are rational. Hence by Newton's identities all of the elementary symmetric functions are also rational. But they are algebraic integers, directly from their definition. Hence they are rational integers.

Suppose that k is an odd non-square. Then every power sum is of the form $k^{r/2}$ times a rational number. Thus every elementary symmetric function is likewise of that form, and at least one of them must contain an uncancelled factor of $\sqrt{k}$. By Newton's identities, the first coefficient of the polynomial $f_k(x)$ that involves that elementary symmetric function will likewise contain a factor $\sqrt{k}$.

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