Sums of closed form functions satisfy recurrence relations

Herbert S. Wilf[1]
University of Pennsylvania
Philadelphia, PA 19104-6395
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Abstract

We give a short, self-contained proof of the fact that sums of closed form (hypergeometric) functions satisfy recurrence relations with polynomial coefficients. We obtain explicit bounds for the order of the recurrence and for the degrees of the coefficient polynomials directly from the form of the summand.

It is by now well known that the title of this paper is a true statement. The pioneering steps in the recognition of this fact, and in the techniques for proving it are due to Sister Mary Celine Fasenmyer in her thesis at Michigan in 1945, and in [1],[2]. For a very readable account of her work see [3], chapter 14.

In this paper we will give a short and entirely self-contained proof in the case of closed form summands. This is certainly not the first proof. It is failry simple, though. Furthermore we give quite an explicit estimate for the order of the resulting recurrence formula, directly from the form of the summand.

The following notation will be helpful. First, the rising and falling factorials will be $x^{\overline{n}}$ and $x^{\underline{n}}$, respectively. Next, define for integer u,

$$A_u(a) = \begin{cases} 1, & \text{if } u = 0; \\ (a+1)^{\overline{u}}, & \text{if } u > 0; \\ 1/a^{\underline{|u|}}, & \text{if } u < 0. \end{cases}$$

This Λ_u is 'really' just $(u+a)!/a!$, rewritten to emphasize that it is a polynomial in a of degree u, if $u \geq 0$, or the reciprocal of a polynomial of degree $|u|$, if $u < 0$. We will write x^+ for $\max(x, 0)$, and $A = \sum a_s, B = \sum b_s, U = \sum u_s, V = \sum v_s$. In the sequel, sums over k will always be understood to run over all integers, and sums over i, j will run over 0 to I and from 0 to J, respectively, where I and J are to be determined.

Consider the closed form function

$$F(n,k) = \frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}$$

and write $f_n(x) = \sum_k F(n,k)x^k$.

Theorem. *The (most general hypergeometric) polynomials $\{f_n(x)\}$ satisfy a recurrence relation of the form*

$$\sum_{j=0}^{J}\sum_{i=0}^{I} \alpha_{ij} x^i f_{n-j}(x) = 0. \qquad (\textit{integer } n)$$

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in which the order of the recurrence, J, is at most

$$J^* = \sum(-v_s)^+ + \sum b_s^+ + (V-B)^+,$$

and the degrees of the polynomials in x that occur as coefficients in the recurrence are at most

$$I^* = J^*\{\sum a_s^+ + \sum(-u_s)^+ + (U-A)^+ - 1\} + 1.$$

To prove this, we have first

$$\begin{aligned}\sum_{i,j} \alpha_{ij} x^i f_{n-j}(x) &= \sum_{i,j} \alpha_{ij} \sum_k F(n-j,k) x^{k+i} \\ &= \sum_{i,j} \alpha_{ij} \sum_k x^k F(n-j,k-i) \\ &= \sum_k x^k F(n,k) \left\{ \sum_{i,j} \alpha_{i,j} \frac{F(n-j,k-i)}{F(n,k)} \right\}.\end{aligned} \tag{1}$$

The quantity inside the large curly braces will be called Γ_k. We want to determine the $\alpha's$, if possible, so that they are independent of k, and so that Γ_k vanishes identically in k. Now,

$$\frac{F(n-j,k-i)}{F(n,k)} = \frac{\prod_{s=1}^p \Lambda_{-ja_s-ib_s}(a_s n + b_s k + c_s)}{\prod^{q_{s=1}} \Lambda_{-ju_s-iv_s}(u_s n + v_s k + w_s)} \tag{2}$$

The plan is to put all terms in Γ_k over a common denominator, and then consider the degree of the numerator as a polynomial in k. Now every denominator that occurs in Γ_k is a divisor of the polynomial

$$\phi(k) = \{\prod_{s=1}^p \Lambda_{f(s)}(a_s n + b_s k + c_s)\}\{\prod_{s=1}^q \Lambda_{g(s)}(u_s n + v_s k + w_s)\}$$

where

$$f(s) = \max\{0, \max_{i,j}(ja_s + ib_s)\} = I(b_s)^+ + J(a_s)^+$$

$$g(s) = \max\{0, \max(-ju_s - iv_s)\} = I(-v_s)^+ + J(-u_s)^+.$$

Hence the common denominator ϕ is a polynomial in k of degree

$$\Delta = I\{\sum_{s=1}^p (b_s)^+ + \sum_{s=1}^q (-v_s)^+\} + J\{\sum_{s=1}^p (a_s)^+ + \sum_{s=1}^q (-u_s)^+\}$$

and every term in Γ_k is a polynomial divided by $\phi(k)$.

We need a bound on the degrees of the numerator polynomials. The numerator of a fixed term $F(n-j,k-i)/F(n,k)$ in (2) is of degree

$$\nu_{ij} = \sum_{s=1}^p (-ja_s - ib_s)^+ + \sum_{s=1}^q (ju_s + iv_s)^+$$

and its denominator is of degree

$$\delta_{ij} = \sum_{s=1}^{p}(ja_s + ib_s)^+ + \sum_{s=1}^{q}(-ju_s - iv_s)^+.$$

Hence when all terms in Γ_k are put over the common denominator $\phi(k)$, the term (i, j) will contribute a polynomial of degree

$$\Delta - \delta_{ij} + \nu_{ij} = \Delta - (A - U)_j - (B - V)_i$$

in k to the common numerator. Thus the numerator of the entire sum is of degree at most

$$\Delta + \max_{i,j}(\nu_{ij} = \Delta + (U - A)^+ J + (V - B)^+ I,$$

as a polynomial in k. If there are more coefficients α_{ij} at our disposal than there are powers of k in the numerator polynomial, i.e., if

$$(I + 1)(J + 1) > \Delta + J(U - A)^+ + I(V - B)^+ + 1,$$

then we can find a nontrivial recurrence relation. This inequality is surely satisfiable if we take a recurrence of order $\geq J^*$, where J^* is as stated in the theorem. $\square$

Example 1: Take $F(n, k) = \binom{n}{k}^2$. Then we are looking for a recurrence formula for the polynomials $f(n, k) = \binom{n}{k}^2 x^k$. Here we have $p = 2$, $a_1 = a_2 = 1$, $b_1 = b_2 = c_1 = c_2 = 0$ and $q = 4$, with $u_1 = u_2 = 0$, $u_3 = u_4 = 0$, $u_3 = u_4 = 1$, and $v_1 = v_2 = 1$, $v_3 = v_4 = -1$, and all $w_i = 0$. Then $\nu_{ij} = 2\max(i, j) = \delta_{ij}$ and $\Delta = 2(I + J)$. Hence there is a recurrence if $(I + 1)(J + 1) > 2(I + J) + 1$, i.e., if $IJ > I + J$. This cannot be satisfied with $J = 1$, but there is a solution with $J = 2$ and $I = 3$. That is, there is a second order recurrence with at most third degree polynomial-in-x coefficients. In fact it is well known that $f_n(x) = (x - 1)^n P_n((x + 1)/(x - 1))$, where the P's are the Legendre polynomials. Hence these f_n's actually satisfy a second order recurrence with quadratic coefficients, namely

$$nf_n(x) = (2n - 1)(x + 1)f_{n-1}(x) - (n - 1)(x - 1)^2 f_{n-2}(x).$$

Example 2: Now fix a positive integer q and put $F(n, k) = \binom{n}{k}^q$. Then computation directly from the statement of the theorem shows that the polynomials $f_n(x; q) = \sum_k \binom{n}{k}^q x^k$ satisfy a recurrence of order q whose coefficients are polynomials in x of degrees at most $q^2 - q + 1$.

Example 3: If $F(n, k) = (n + k + \alpha + \beta)!/(k!(n - k)!(k + \alpha)!)$ then the f_n's become the Jacobi polynomials. Our theorem asserts that these satisfy a recurrence of order $J^* = 2$ whose coefficients are linear in x, which in this case is the best possible result.

The proof is quite constructive. To write a program, in a language with symbolic manipulation capability, that will find the recurrence relations, one first computes the rational function Γ_k in the curly brace in (1). Next collect the terms over a common denominator and equate to zero the coefficient of each power of k that occurs in the numerator. If I and J are chosen as stated in the theorem, then these homogeneous equations in the coefficients α_{ij} will have a nontrivial solution, and we will have found the desired recurrence.

Finally, it would be desirable to have tight bounds for the degrees of the coefficient polynomials when regarded as polynomials in n. If we let $M = (I^* + 1)(J^* + 1)$, then the coefficients of the recurrence lie in the kernel of an $(M + 1) \times M$ matrix whose entries are polynomials in n of degrees at most $M - 1$. Thus by Cramer's rule, no terms of degree higher than $n^{M(M-1)}$ can appear. This is likely to be a considerable overestimate, however.

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