The asymptotic behavior of the Stirling numbers of the first kind

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1. Results

The Stirling numbers of the first kind, $\left[{n \atop k}\right]$, have a lovely asymptotic expansion for fixed k, as $n \to \infty$. It involves the power series coefficients of $1/\Gamma(z)$ about $z = 0$, where Γ is the classical gamma function. We state the result immediately insofar as the terms that do not approach zero are concerned. Later we will discuss the complete development.

Theorem 1. *If $\left[{n \atop k}\right]$ is the (signless) Stirling number of the first kind, then for each fixed integer $k > 1$ we have*

$$\frac{1}{(n-1)!}\left[{n \atop k}\right] = \gamma_1 \frac{(\log n)^{k-1}}{(k-1)!} + \gamma_2 \frac{(\log n)^{k-2}}{(k-2)!} + \cdots + \gamma_k + O\left(\frac{(\log n)^{k-2}}{n}\right)$$

where the γ_j are the coefficients in the expansion

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \sum_{j=1}^{\infty} \gamma_j z^j \\ &= z + \gamma z^2 + \frac{(6\gamma^2 - \pi^2)}{12} z^3 + \frac{\left(2\gamma^3 - \gamma\pi^2 + 4\zeta(3)\right)}{12} z^4 \\ &\qquad + \frac{\left(60\gamma^4 - 60\gamma^2\pi^2 + \pi^4 + 480\gamma\zeta(3)\right)}{1440} z^5 + \cdots \\ &= z + 0.5772156649015..z^2 - 0.6558780715203..z^3 - 0.04200263503410..z^4 \\ &\qquad + 0.1665386113823..z^5 + \cdots\end{aligned}$$

and γ is Euler's constant.

In this paper we will first prove the theorem above, then extend it to a form from which one can find as many terms of the expansion as desired, and finally discuss the relationship of our work to earlier results of Moser and Wyman [3].

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2. Proof of theorem 1

We will write $\zeta_n(s)$ for $\sum_{j=1}^{n} j^{-s}$, which, if $s > 1$, is the nth partial sum of the Dirichlet series for the Riemann zeta function. The following computation will be the basis for the developments in the sequel. We have, from the familiar generating function,

$$\begin{aligned}\frac{1}{n!}\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^k &= (1+x)(1+\frac{x}{2})\cdots(1+\frac{x}{n}) \\ &= \exp\{\sum_{j=1}^{n} \log(1+\frac{x}{j})\} \\ &= \exp\{\sum_{j=1}^{n}\sum_{s\geq 1}(-1)^{s-1}\frac{x^s}{sj^s}\} \\ &= \exp\{\sum_{s\geq 1}(-1)^{s-1}\frac{x^s\zeta_n(s)}{s}\} \\ &= \prod_{s=1}^{\infty}\exp\{(-1)^{s-1}\frac{x^s\zeta_n(s)}{s}\}.\end{aligned} \tag{1}$$

In the last member we wish to replace all of the ζ_n's by their asymptotic expansions in powers of n and $\log n$. These expansions are (e.g., [5], p. 124)

$$\zeta_n(s) \approx \begin{cases} \log n + \gamma + \sum_{j=1}^{\infty} d(1,j)n^{-j}, & \text{if } s = 1; \\ \zeta(s) + (1-s)^{-1}n^{-s+1} + \sum_{j=1}^{\infty} d(s,j)n^{-s-j+1}, & \text{if } s > 1, \end{cases} \tag{2}$$

where

$$d(s,j) = -\frac{B_j}{j}\binom{s+j-2}{j-1}$$

and the B_j's are the Bernoulli numbers.

By combining (1) and (2), we now have

$$\frac{1}{n!}\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^k \approx \exp\left\{x(\log n + \gamma + \sum_{j=1}^{\infty}\frac{d(1,j)}{n^j})\right\}\times$$
$$\prod_{s\geq 2}\exp\left\{(-1)^{s-1}\frac{x^s}{s}\{\zeta(s) + \frac{1}{(1-s)n^{s-1}} + \sum_{j=1}^{\infty}\frac{d(s,j)}{n^{s+j-1}}\}\right\}, \tag{3}$$

in the sense that for each fixed k, the coefficient of x^k on the left has, for its complete asymptotic series as $n \to \infty$, the coefficient of x^k on the right.

Now we can prove theorem 1. To do that we need only keep the portion of the development in powers of n and $\log n$ that does not approach zero on the right side of (3), along with the size of the first neglected term. If we take the coefficient of x^k on both sides of (3) we obtain

$$\frac{1}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \mathrm{Coeff}_{\{x^k\}}\{e^{x(\log n+\gamma)} \exp\{\sum_{s\geq 2}(-1)^{s-1}\frac{x^s\zeta(s)}{s}\}\} + O(\frac{(\log n)^{k-1}}{n}).$$

The second exponential factor above is quite interesting, however. It is

$$\exp\{\sum_{s\geq 2}(-1)^{s-1}\frac{x^s\zeta(s)}{s}\} = \exp\{\sum_{s\geq 2}(-1)^{s-1}\frac{x^s}{s}\sum_{j\geq 1}j^{-s}\} = \prod_{j\geq 1}(1+\frac{x}{j})e^{-x/j} = \frac{e^{-\gamma x}}{x\Gamma(x)}. \quad (4)$$

Hence

$$\frac{1}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \mathrm{Coeff}_{\{x^{k+1}\}}\{\frac{e^{x\log n}}{\Gamma(x)}\} + O(\frac{(\log n)^{k-1}}{n})$$

which completes the proof of theorem 1.

3. Extensions

In this section we describe how to find as many terms of the asymptotic expansion of $\left[{n \atop k}\right]$ as desired. As an illustration we will state the expansion with an error of $O(n^{-3+\epsilon})$.

To do this it will be helpful to define polynomials $\{v_k(y)\}_{k=0}^{\infty}$ by

$$\frac{e^{xy}}{x\Gamma(x)} = \sum_{k=0}^{\infty} v_k(y)x^k. \quad (5)$$

Thus, $v_0(y) = 1$, $v_1(y) = y+\gamma$, $v_2(y) = (6y^2 + 12\gamma y + 6\gamma^2 - \pi^2)/12$, ..., and in general $v_k(y) = \sum_{r=0}^{k} \gamma_{k+1-r}y^r/r!$. Now we can rewrite (3), using (4), as follows.

Theorem 2. *The complete asymptotic series for the Stirling numbers of the first kind, for fixed k, is obtained by equating the coefficients of x^k on both sides of the relation*

$$\frac{1}{n!}\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^k \approx \{\frac{e^{x\log n}}{x\Gamma(x)}\}\exp\left\{\sum_{j=1}^{\infty}\frac{\psi_j(x)}{n^j}\right\}, \quad (6)$$

where the polynomials $\psi_j(x)$ are given by

$$\psi_j(x) = \frac{(-x)^{j+1}}{j(j+1)} + \sum_{s=1}^{j}\binom{j-1}{s-1}\frac{B_{j+1-s}(-x)^s}{s(j+1-s)} \qquad (j=1,2,\ldots).$$

Then the asymptotic expansion of $\left[{n+1 \atop k+1}\right]/n!$ can be obtained through terms of order n^{-q}, say, as follows.

(i) Expand the first factor on the right of (6) in powers of x with error $O(x^{k+1})$, using (5).

(ii) Expand the second factor on the right of (6) in powers of $1/n$, with error $O(1/n^{q+1})$. In this expansion, drop all powers of x that are higher than x^k.

(iii) Find the coefficient of x^k.

For example, with error $O(n^{-3})$ we find that

$$\frac{1}{n!}\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^k = \{\frac{e^{x\log n}}{x\Gamma(x)}\}\{1+(\frac{1}{2n}-\frac{1}{12n^2})x+(\frac{1}{2n}-\frac{1}{8n^2})x^2+\frac{x^3}{12n^2}+\frac{x^4}{8n^2}+O(n^{-3})\},$$

which translates at once into the following refinement of Theorem 1.

$$\begin{aligned}\frac{1}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} &= \sum_{r=0}^{k}\frac{(\log n)^r}{r!}\{\gamma_{k+1-r}+(\frac{1}{2n}-\frac{1}{12n^2})\gamma_{k-r}+(\frac{1}{2n}-\frac{1}{8n^2})\gamma_{k-r-1} \\ &\qquad +\frac{1}{12n^2}\gamma_{k-r-2}+\frac{1}{8n^2}\gamma_{k-r-3}\}+O(n^{-3+\epsilon}) \\ &= \sum_{r=0}^{k}\frac{(\log n)^r}{r!}\{\gamma_{k+1-r}+\frac{1}{2n}(\gamma_{k-r}+\gamma_{k-r-1}) \\ &\quad +\frac{1}{24n^2}(-2\gamma_{k-r}-3\gamma_{k-r-1}+2\gamma_{k-r-2}+3\gamma_{k-r-3})\}+O(n^{-3+\epsilon}).\end{aligned} \tag{7}$$

4. History and remarks

The Stirling numbers of the first kind are notoriously difficult to compute. Unlike the numbers of the second kind, the closed formulas for $\left[{n \atop k}\right]$ whose summands are the familiar factorials, binomial coefficients, etc., involve two summation signs. The first asymptotic formula for them was given by Jordan [2], who showed, for k fixed, that

$$\frac{1}{(n-1)!}\begin{bmatrix} n \\ k \end{bmatrix} \sim \frac{(\log n+\gamma)^{k-1}}{(k-1)!}. \tag{8}$$

Moser and Wyman [3] gave a comprehensive study of the asymptotics of $\left[{n \atop k}\right]$ for large n. Further, their study was not restricted to fixed k, but indeed covered the entire range of $1 \le k \le n$ by breaking it into three subranges and dealing separately with each of them. The formulas that they obtain for the low end of the range, i.e., for fixed k, are however not phrased in terms of the usual atoms of asymptotics, namely powers of n and of $\log n$. Instead the formulas are expressed as series of polynomials in the partial sums of the zeta function, containing $p(k)$ monomials, $p(k)$ being the number of partitions of k. Using our present notation, their formula for the range of small k begins as

$$\begin{aligned}\frac{1}{(n-1)!}\left[{n \atop k}\right] = \frac{\zeta_n(1)^{k-1}}{(k-1)!}\Big\{1 - \binom{k-1}{2}\frac{\zeta_n(2)}{\zeta_n(1)^2} + 2\binom{k-1}{3}\frac{\zeta_n(3)}{\zeta_n(1)^3} \\ + 3\binom{k-1}{4}\frac{\zeta_n(2)^2 - 2\zeta_n(4)}{\zeta_n(1)^4} + \cdots\Big\}.\end{aligned} \tag{9}$$

Our formulas above are considerably more explicit, though they have been proved here to hold only for constant k, and in no event can their validity extend beyond about $k \sim C \log n / \log\log n$.

Two numerical examples are shown in the following table.

	$\left[{100 \atop 5}\right]/99!$	Error	$\left[{50 \atop 9}\right]/49!$	Error
Exact value . . .	21.1204415..	$--$	0.5591960..	$--$
Jordan formula . .	18.740..	9.9%	4.09..	700%
3 terms of eq.(9) .	21.24986..	0.612%	0.95816..	71%
Theorem 1 above .	20.960..	0.76%	0.5076466..	9.2%
Eq. (7) to order $1/n$	21.12070..	0.0012%	0.558663..	0.095%
Eq. (7) to order $1/n^2$	21.1204409..	0.000003%	0.5591994..	0.0003%

The polynomials $\{v_k(y)\}$ that are defined by (5) above are a polynomial sequence of Sheffer A-type zero ([4], pp. 221-226), and consequently a number of recurrence formulas, index-raising operators, etc., are available. The first several of these polynomials appear to have only real zeros, though I am not aware of any theorems that would assure this.

The computation of the power series coefficients γ_j of the expansion of $1/\Gamma(z)$ can be quite time-consuming. It can be speeded up considerably by using the recurrence

$$\gamma_{n+1} = \frac{1}{n}\Big\{\gamma\gamma_n + \sum_{j=0}^{n-2}(-1)^{n-j-1}\zeta(n-j)\gamma_{j+1}\Big\} \qquad (n \ge 1;\ \gamma_1 = 1),$$

that comes from logarithmic differentiation of the infinite product. The asymptotic sign fluctuations in the sequence $\{\gamma_j\}$ have been studied by Hayman [1].

References

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