The Rectilinear Crossing Number of a Complete Graph and Sylvester's "Four Point Problem" of Geometric Probability

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December 16, 1994

"The chance of ... the quadrilateral formed by joining four points, taken arbitrarily within any assigned boundary, constituting a reentrant or convex quadrilateral, will serve as types of the class of questions in view."

—J.J. Sylvester [11]

We prove that two fundamental constants of the geometry of the plane are equal.

First, if R is an open set in the plane with finite Lesbesgue measure, let $q(R)$ denote the probability that if four points are chosen independently uniformly at random in R, then their convex hull is a quadrilateral. Let q_* be the infimum of $q(R)$ over all such sets R.

Second, let $\bar{\nu}(K_n)$ denote the rectilinear crossing number of the complete graph on n vertices, i.e., the minimum number of intersections in any drawing of K_n in the plane that has straight-line-segment edges. It is well known that $\bar{\nu}(K_n)/\binom{n}{4}$ increases steadily to some limit $\bar{\nu}^*$ as $n \to \infty$.

Our main result is that $q_* = \bar{\nu}^*$.

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Four random points

Let R be an open set in the plane with finite area. As such, we can consider R to be a sample space from which we select points independently uniformly at random (i.u.a.r.). Choose four points from R i.u.a.r.. Then with probability 1, no three of the points are collinear, so the convex hull of the four points is either a triangle (one point in the convex hull of the other three) or a quadrilateral. J.J. Sylvester [11] asked, what is the probability that the points determine a convex quadrilateral? We denote this probability by $q(R)$.

How large and how small can $q(R)$ be? When R is restricted to being a convex set we have the following result (see Blaschke [1, 2] and also [7]).

Theorem 1 *Let R be an open, convex subset of the plane, of finite area.*

Then

$$\frac{2}{3} \leq q(R) \leq 1 - \frac{35}{12\pi^2} \approx 0.704.$$

Further, both inequalities are sharp. The lower bound is attained by the interior of a triangle (any triangle), and the upper bound by the interior of an ellipse (any ellipse). □

This theorem, however, does not fully address the issue of the extreme values of $q(R)$ because it considers only *convex* regions R. It is easy to see that if we relax the convexity requirement, then the supremum of $q(R)$ is 1; let R be a very thin open annulus and observe that we can make $q(R)$ arbitrarily close to 1. Thus it remains to consider the infimum of $q(R)$; let $q_* = \inf q(R)$ where the infimum is over all open sets R with finite area.

We show below that q_* is positive and strictly less than 2/3 (the lowest possible result for convex R). We show further that q_* is closely related to the rectilinear crossing number of complete graphs.

Rectilinear crossing number of a graph

Let G be a graph which we wish to draw in the plane. If G is planar, then we can find an embedding in which the edges do not cross. A result of Fáry [4] shows that we can choose this embedding so that the edges are noncrossing straight line segments.

Let $\bar{\nu}(G)$ denote the minimum number of crossings in a straight line drawing of G in the plane; the parameter $\bar{\nu}(G)$ is known as the *rectilinear crossing number* of G. (For background on the rectilinear crossing number, see [6], [7] or [12].)

An important open problem in the study of graph embeddings is to determine the rectilinear crossing number of the complete graph K_n. For $n = 5, 6, 7, 8, 9$ the values are known (see [12]) and they are $1, 3, 9, 19, 36$ respectively. For $n = 10$ it is known [10] that $61 \leq \bar{\nu}(K_{10}) \leq 62$.

If we place the n vertices of K_n on a circle, then the number of crossings is exactly $\binom{n}{4}$; certainly we can do better, but $\bar{\nu}(K_n)$ is on the order of n^4 as we now explain.

Theorem 2 *There exists a constant $\bar{\nu}^*$ such that $0 < \bar{\nu}^* < \infty$ and*

$$\bar{\nu}^* = \lim_{n\to\infty} \frac{\bar{\nu}(K_n)}{\binom{n}{4}} = \sup_n \frac{\bar{\nu}(K_n)}{\binom{n}{4}}.$$

(This is well-known folklore, but for completeness we show the proof here.)

Proof. Let $m < n$ and consider a straight line embedding of K_n in the plane with the minimum number of crossings, $\bar{\nu}(K_n)$. For each m element subset A of $V(K_n)$, let $c(A)$ denote the number of crossings in this embedding in which the endpoints of the crossing edges are all in A. If we sum $c(A)$ over all m-subsets of $V(K)$, we count each possible crossing exactly $\binom{n-4}{m-4}$ times. Thus

$$\bar{\nu}(K_n) = \sum_{|A|=m} c(A) \Big/ \binom{n-4}{m-4}$$

Now clearly $c(A) \geq \bar{\nu}(K_m)$, so it follows that

$$\bar{\nu}(K_n) \geq \frac{\binom{n}{m}}{\binom{n-4}{m-4}} \bar{\nu}(K_m)$$

which we can rearrange to read

$$\frac{\bar{\nu}(K_n)}{\binom{n}{4}} \geq \frac{\bar{\nu}(K_m)}{\binom{m}{4}}.$$

Thus $\bar{\nu}(K_n)/\binom{n}{4}$ is a nondecreasing function of n which is bounded above by 1 and below by $\bar{\nu}(K_5)/\binom{5}{4} = 1/5$. $\square$

Since $\bar{\nu}(K_{10}) \geq 61$, we see that $\bar{\nu}^* \geq 61/210 \approx 0.29$.

Singer [10] proves the following upper bound on the rectilinear crossing number of K_n when n is a power of 3:

$$\bar{\nu}(K_n) \leq \frac{1}{312}\left(5n^4 - 39n^3 + 91n^2 - 57n\right).$$

Thus $\bar{\nu}^* \leq \frac{5}{312} \times 24 = \frac{5}{13} \approx 0.3846$.

(Jensen [6] gives a rectilinear embedding of K_n with $\frac{7}{432}n^4 + O(n^3)$ crossings, yielding an upper bound of $0.3888\ldots$ on $\bar{\nu}^*$.)

Main result

Our main result is a simple relation between q_*, the smallest probability of choosing a quadrilateral, and $\bar{\nu}^*$, the limit of $\bar{\nu}(K_n)/\binom{n}{4}$.

Theorem 3 *With the preceding notation, $q_* = \bar{\nu}^*$.*

Proof. Let R be any open set in the plane with finite area. Choose n points $p_1, p_2, \ldots, p_n$ i.u.a.r. from R and let those n points be the vertices of a straight line drawing of K_n. Let c be the number of crossings in this drawing. Now c is a random variable whose value is always at least $\bar{\nu}(K_n)$. Further,

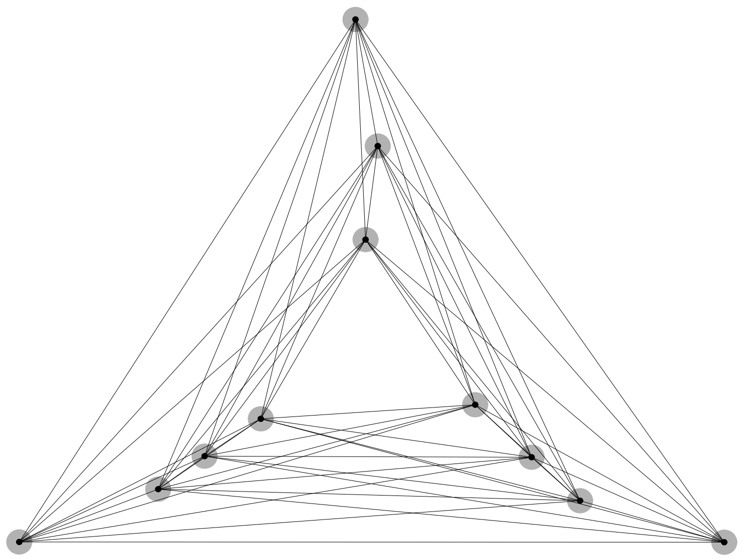
$$X = \sum_{\{a,b,c,d\}} \mathbf{1}\{p_a, p_b, p_c, p_d \text{ form a quadrilateral}\}$$

where the sum is over all 4 element subsets of $\{1, \ldots, n\}$ and $\mathbf{1}\{\ldots\}$ is a $0,1$ indicator random varible whose value is 1 just when the convex hull of the four points p_a, p_b, p_c, p_d is a quadrilateral. Since the optimum drawing cannot have more crossings than the average, we get, by taking expectations,

$$\bar{\nu}(K_n) \leq E(X) = \binom{n}{4} q(R)$$

for all n. Dividing by $\binom{n}{4}$ and letting $n \to \infty$, we have $\bar{\nu}^* \leq q(R)$ for all R. Thus $\bar{\nu}^* \leq q_*$.

For the opposite inequality, consider a straight line embedding of K_n with the minimum number, $\bar{\nu}(K_n)$, of crossings. Let R_ϵ be the (disconnected) open set formed by placing a small open disc of radius ϵ centered at each vertex of the embedding. The figure below shows this in the case $n = 11$.



Here ϵ is chosen small enough so that for every choice of n points, one in each disc, if we connect all pairs of them by straight line segments then the number of crossings is always equal to $\bar{\nu}(K_n)$, i.e., all such embeddings are optimal. Clearly such an ϵ exists.

We now consider the following question: choose four distinct discs of R_ϵ, and then choose i.u.a.r. a point from each of them. What is the probability q that the resulting quadrilateral is convex?

On the one hand, q is the number of convex quadrilaterals in the original embedding divided by $\binom{n}{4}$. But the former are in 1-1 correspondence with edge crossings, so there are exactly $\bar{\nu}(K_n)$ of them, and we have $q = \bar{\nu}(K_n)/\binom{n}{4}$.

On the other hand, $q(R_\epsilon)$ is the probability that four points chosen i.u.a.r. in R_ϵ will form a convex quadrilateral. But four points so chosen will lie in four distinct discs of R_ϵ with probability $1 - O(1/n)$. Hence

$$q = q(R_\epsilon) + O(1/n) \geq q_* + O(1/n).$$

Combining these two facts about q we obtain

$$q = \bar{\nu}(K_n) / \binom{n}{4} \geq q_* + O(1/n).$$

If we now let $n \to \infty$, $\bar{\nu}^* \geq q_*$ follows, and the proof is complete. $\square$

Thus to summarize our principal results, we have

$$0.29 \approx \frac{61}{210} \leq q_* = \bar{\nu}^* \leq \frac{5}{13} \approx 0.385.$$

Some comments

1. In a spirit similar to ours, Moon [8] applies random methods in bounding the ordinary crossing number of K_n. He places n points i.u.a.r. on a sphere and joins them pairwise by arcs of great circles. This gives an upper bound of $\frac{3}{8}\binom{n}{4}$ for the crossing number of K_n. It is not clear how to project Moon's embedding into the plane and have the edges become line segments.

2. Our methods can be applied to arbitrary graphs G. Let M denote the number of pairs of edges in G which span four distinct vertices. Then $\bar{\nu}(G) \leq \bar{\nu}^* M/3$; simply place the vertices of G i.u.a.r. in a region R and compute the expected number of crossings.

3. Given a subset R of the plane together with a probability measure μ defined on R, define $q(R, \mu)$ to be the probability that four points chosen i.u.a.r. with respect to μ form a convex quadrilateral. Without further restrictions, we see that the infimum of $q(R, \mu)$ is 0; let R be a unit line segment together with Lebesgue measure—no four points can form a quadrilateral.

   If we restrict ourselves to those (R, μ) for which the probability that three points selected i.u.a.r. are collinear is 0, then the infimum of $q(R, \mu)$ remains the same, namely q_*.

   If we understand Sylvester's problem (see quote above) to mean the interior of a Jordan curve, our results still don't change; in the proof of Theorem 3 we can join the small open disks by even smaller tendrils so the domain is the interior of a simple closed curve.

4. Let R be a bounded open set and embed the vertices of K_n at n points selected i.u.a.r. in R. We have seen that the expected number of crossings in this embedding is $q(R)\binom{n}{4}$. However, one might harbor hopes of doing better on occasion. It would seem natural to generate many embeddings of K_n in, say, the interior of a square or a disk, and count the number of crossings in hopes of finding a good embedding. Regrettably, this is not at all likely, as we now explain.

   The number of crossings can be written

$$X = \sum_{\{a,b,c,d\}} \mathbf{1}\{p_a, p_b, p_c, p_d \text{ form a quadrilateral}\}$$

   where the sum is over all 4-element subsets of $\{1, \ldots, n\}$ and the p_i's are chosen i.u.a.r. in R. Thus X is an example of a *U-statistic*; see [5] or [9] for an extensive discussion. Using "deviations" results (see [9] §5.6) one can show that the probability that the number of crossings is "significantly" less than the expectation is extremely small.

References

[1] Blaschke, W., Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der Theorie der geometrischen Wahrscheinlichkeiten, *Leipziger Berichte* **69** (1917) 436–453.

[2] Blaschke, W., *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Springer, Berlin (1923).

[3] Erdős, P. and R.K. Guy, Crossing number problems, This *Monthly* **80** (1973) 52–57.

[4] Fáry, I., On straight line representations of planar graphs, *Acta Sci. Math. (Szeged)* **11** (1948) 229–233.

[5] Hoeffding, W., The strong law of large numbers for U-statistics, Univ. North Carolina Institute of Statistics Mimeo Series, no. 302 (1961).

[6] Jensen, H.F., An upper bound for the rectilinear crossing number of the complete graph, *J. Comb. Theory (B)* **10** (1971) 212–216.

[7] Klee, Victor, What is the expected volume of a simplex whose vertices are chosen at random from a given convex body?, This *Monthly* **76** (1969) 286–288.

[8] Moon, J.W., On the distribution of crossings in random complete graphs, *SIAM J.* **13** (1965) 506–510.

[9] Serfling, Robert J., *Approximation Theorems of Mathematical Statistics*, Wiley (1980).

[10] Singer, David, The rectilinear crossing number of certain graphs, manuscript (1971).

[11] Sylvester, J.J., On a special class of questions on the theory of probabilities, *Birmingham British Association Report* (1865) 8.

[12] White, Arthur T. and Lowell W. Beineke, Topological graph theory, in *Selected Topics in Graph Theory*, Beineke & Wilson, eds., Academic (1978) 15–49.