Knot Homology from Refined Chern-Simons Theory

Mina Aganagic

UC Berkeley

Based on work with Shamil Shakirov

arXiv: 1105.5117

Witten explained in '88 that

the knot invariant $J(K,q)$ constructed by Jones

$$q^2 + q^6 - q^8$$

$$q^6 + q^{10} - q^{16}$$

is computed by SU(2) Chern-Simons theory on S^3.

Chern-Simons theory with gauge group G on a three-manifold M is given in terms of the path integral

$$Z_{CS}(M) = \int \mathcal{D}A \, e^{\frac{ik}{4\pi} S_{CS}(\mathcal{A})}$$

where A is a connection valued in the Lie algebra of G

k is an integer, and the action is

$$S_{CS} = \int_M \mathrm{Tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big)$$

The theory is independent of the metric on M, so is automatically topologically invariant.

The natural observable in the theory is the Wilson loop,

$$\mathcal{O}_R(K) = \mathrm{Tr}_R P \exp i \oint_K A$$

associated to a knot K in M

taken in various representations R of the gauge group.

To any three manifold with a collection of knots in it,

the Chern-Simons path integral

$$Z_{CS}(M; K_1, \ldots, K_n) = \int \mathcal{D}A \, e^{\frac{ik}{4\pi} S_{CS}(\mathcal{A})} \mathcal{O}_{R_1}(K_1) \ldots \mathcal{O}_{R_n}(K_n)$$

associates a knot and three manifold invariant.

The Jones polynomial $J(K,q)$ is obtained from Chern-Simons theory by taking G=SU(2),

and R_i the fundamental, two-dimensional representation.

More generally, the expectation value of Wilson loop in fundamental representation of G=SU(N) computes the HOMFLY polynomial $H(K,q,a)$, at $a = q^N$

Jones' construction relied on projections of knots to two dimensions. While one could prove that the invariant is independent of the projection chosen, the relation to Chern-Simons theory made it manifest that one obtains invariants of knots in three-dimensional space.

Moreover, Chern-Simons theory is solvable explicitly, so it provides a physical way to not only interpret the knot polynomial, but to calculate it as well.

While Chern-Simons theory provides a way to compute the knot invariants, it does not explain why one obtains polynomials in q,

(or q and a) with integer coefficients.

A way to answer this question was provided by Khovanov in '99.

One associates to a

knot K bi-graded cohomology groups $\mathcal{H}^{i,j}(K)$ in such a way that

the Jones polynomial

$$J(K)(\mathbf{q}) = \sum_{i,j} (-1)^i \mathbf{q}^j \dim \mathcal{H}^{i,j}(K)$$

arises as the Euler characteristic,

taken with respect to one of the gradings.

In addition to explaining the integrality of Jones and related knot polynomials, knot homology has more information than its Euler characteristic. In particular, in terms of its ability to distinguish knots the Poincare polynomial

$$Kn(K)(\mathbf{q},\mathbf{t}) = \sum_{i,j} \mathbf{t}^i \mathbf{q}^j \dim \mathcal{H}^{i,j}(K)$$

is a finer invariant.

Like Jones's theory, Khovanov's theory also relies on two-dimensional projections of knots, so again one wants to have a three or higher dimensional framework for formulating and computing knot homologies.

A natural way to do this would interpret knot homology groups as Hilbert spaces of a theory in one higher dimensions, as a part of the categorification program.

I will explain that, if one is less general, and restricts to three manifolds and knots admitting a (semi-)free $U(1)$ action, there is way to formulate a three dimensional topological theory, which is a refinement of Chern-Simons theory and computes Poincare polynomials of a knot homology theory categorifying the HOMFLY polynomial.

The refined Chern-Simons theory with gauge group

$G = SU(N)$

is formulated using the

"refined" topological string and M-theory.

Three manifolds which admit a (semi-)free $U(1)$ action are Seifert manifolds, which are circle fibrations over a Riemann surface.

Moreover, just like Chern-Simons theory,

the refined Chern-Simons theory is solvable explicitly.

To begin with, recall the relation of the ordinary topological string to Chern-Simons theory, and the Jones polynomial.

Consider the ordinary A-model topological string on

$$Y = T^*M$$

with N D-branes on a three-manifold M.

The A-model topological string is summing over holomorphic maps from Riemann surfaces with boundaries into Y, where the boundaries fall onto M.

In T^*M, all such maps are degenerate, so open A-model string theory becomes a point particle theory, which is Chern-Simons theory on M:

$$Z_{CS}(M) = Z^{top}_{open}(T^*M)$$

Witten '92

One can include knots K in M,

by adding D-branes on a Lagrangian submanifold

L_K in T^*M , in such a way that

$$L_K \cap M = K$$

Ooguri and Vafa
'99

The open topological string partition function in the presence of D-branes on M and L is computing the Chern-Simons knot invariants associated to the knot K in M and, colored by arbitrary representations of SU(N).

There is an equivalent formulation of the topological A model string on a Calabi-Yau Y using M-theory on

$$(Y \times TN \times S^1)_q$$

where TN is the Taub-Nut space, and as one goes around the S^1, the complex coordinates of TN rotate by

$$z_1 \to q z_1, \qquad z_2 \to q^{-1} z_2$$

Adding N D-branes on a Lagrangian M in $Y = T^*M$

corresponds to adding N M5 branes on

$$(M \times \mathbb{C} \times S^1)_q$$

inside

$$(T^*M \times TN \times S^1)_q$$

where $\mathbb{C}$ corresponds to the $z_2 = 0$ plane in Taub-Nut space.

The partition function of N M5 branes in this geometry

$$Z_{M5}(M,q) = Tr\left(-1\right)^{F} q^{S_1 - S_2}$$

equals the $SU(N)$ Chern-Simons partition function on M

$$Z_{M5}(M,q) = Z_{CS}(M)$$

Here $q = e^{\frac{2\pi i}{k+N}}$ is related to the level k of Chern-Simons

and

S_1, S_2 are generators of rotations around the z_1, z_2 planes of TN.

The refined topological string theory on Y is defined as the M-theory partition function on a slightly more general background,

$$(Y \times TN \times S^1)_{q,t}$$

where as one goes around the S^1, the complex coordinates of TN rotate by

$$z_1 \to q z_1, \qquad z_2 \to t^{-1} z_2$$

To preserve supersymmetry, this has to be accompanied by a $U(1)_R$-symmetry twist. This constrains Y to be non-compact.

Taking $Y = T^* M$, we now add N M5 branes on

$$(M \times \mathbb{C} \times S^1)_{q,t}$$

inside

$$(Y \times TN \times S^1)_{q,t}$$

This preserves supersymmetry
provided M admits a (semi-)free circle action.
Then the $U(1)_R$ symmetry that one needs to twist by as one goes around the circle for general q,t exists.

When the theory on N M5 branes on

$$(M \times \mathbb{C} \times S^1)_{q,t}$$

preserves supersymmetry, its partition function
is the partition function of a topological theory on M,

$$Z_{M5}(M,q,t) = Tr\left(-1\right)^F q^{S_1 - S_R} t^{S_R - S_2}$$

which refines the SU(N) Chern-Simons theory.
Above, S_1, S_2 generate rotations as before,
and S_R generates the R-symmetry twist

The refined Chern-Simons theory is solvable exactly.

Recall that, in a topological theory in three dimensions,
all amplitudes, corresponding to any three-manifold and collection of knots in it, can be written in terms of three building blocks,
the S and T matrices and the braiding matrix B.

In refined Chern-Simons theory, only a subset of amplitudes generated by S and T enter,
since only those preserve the $U(1)$ symmetry.

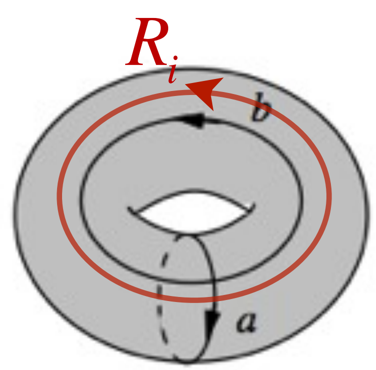
The S and the T matrices of the refined Chern-Simons theory are
a one parameter deformation of the S and T of Chern-Simons theory.

The S and T matrices of a 3d topological quantum field theory, provide a unitary representation of the action of the modular group SL(2, Z)
on the Hilbert space $\mathcal{H}_{T^2}$ of the theory on T.[2]

$$S^4 = 1, \qquad (ST)^3 = S^2.$$

The basis of Hilbert space $\mathcal{H}_{T^2}$ itself is obtained by considering the path integral on a solid torus, with Wilson loops in representations R_i inserted in its interior, along the *b* cycle of the torus.

$$|R_i\rangle \in \mathcal{H}_{T^2}$$



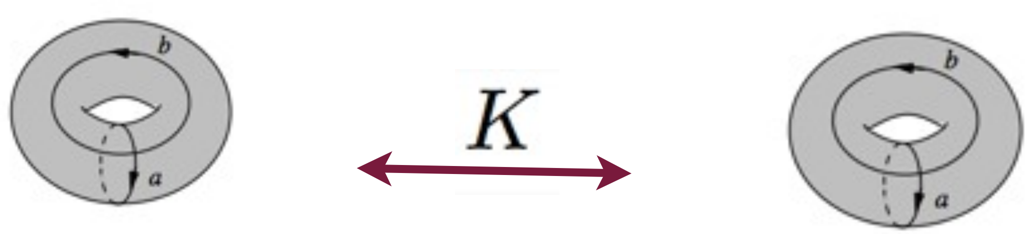
An element K of SL(2,Z) acting on the boundary of the torus, permutes the states in $\mathcal{H}_{T^2}$

$$K|R_i\rangle = \sum_j K^j{}_i |R_j\rangle$$

The matrix element

$$\langle R_j|K|R_i\rangle = K_{\bar{j}i}$$

has interpretation as the path integral on a three manifold obtained by gluing two solid tori,



up to the SL(2,Z) transformation of their boundaries.

In Chern-Simons theory with gauge group G,
the Hilbert space is the space of conformal blocks
of the corresponding WZW model.
The S,T and B are the S,T and B matrices of the WZW model.

This allows one solve the theory explicitly, and in particular
obtain the Jones and other knot polynomials.

Explicitly, for G=SU(N), the S and the T matrices are given by

$$S_{\bar{i}j}/S_{00} = s_{R_i}(q^{\rho})\, s_{R_j}(q^{\rho+\lambda_{R_j}})$$

$$T_{\bar{i},j} = T_i \delta_{\bar{i},j} = c_T q^{\frac{1}{2}(\lambda_{R_i}+\rho,\lambda_{R_i}+\rho)} \delta_{\bar{i},j}$$

$$S_{00} = c_S \prod_{i=1}^{N-1} (q^{i/2} - q^{-i/2})^{N-i}$$

the Hilbert space is labeled by SU(N) representations at level k,

and

$$q = e^{\frac{2\pi i}{k+N}}$$

These expressions are well known. A fact less well known is that one could have discovered them from the topological string and M-theory even without the knowledge of WZW models.

Aganagic, Klemm, Marino, Vafa

Moreover, from M-theory and the refined topological string, one can deduce the S and T matrices of the refined Chern-Simons theory.

In the refined Chern-Simons theory, one finds, for $t = q^{\beta}$, $\beta \in \mathbb{N}$

$$S_{\bar{i}j}/S_{00} = M_{R_i}(t^{\rho})M_{R_j}(q^{\lambda_{R_i}}t^{\rho})$$

$$T_{\bar{i}j} = T_i\delta^i_j = c_T q^{\frac{1}{2}(\lambda_{R_i}+\beta\rho)^2}\delta^i_j$$

$$S_{00} = c_S \prod_{m=0}^{\beta-1}\prod_{i=1}^{N-1}(q^{-m/2}t^{-i/2} - q^{m/2}t^{i/2})^{N-i}$$

the Hilbert space is labeled by SU(N) representations at level k, and

$$q = e^{\frac{2\pi i}{k+\beta N}}$$

The idea of the derivation is to use the M theory definition and topological invariance to solve the theory on the simplest piece, the solid torus

$$M_L = D \times S^1$$

where D is a disk.

S, T and everything else can be recovered from this by gluing.

Using topological invariance, we can replace this by

$$M_L = \mathbf{R}^2 \times S^1$$

and then we are studying M-theory on

$$Y_L = T^* M_L = \mathbb{C}^2 \times \mathbb{C}^*$$

with N M5 branes on M_L.

This is flat space, and M-theory of this is not hard to understand.

In addition, we can separate the branes along the $\mathbb{C}^*$ direction.

Then, the M-theory index

$$Z_{M5}(M,q,t) = Tr\left(-1\right)^F q^{S_1 - S_R} t^{S_R - S_2}$$

can be computed by counting BPS states of M2 branes ending on M5 branes.

Gopakumar and Vafa

Ooguri and Vafa

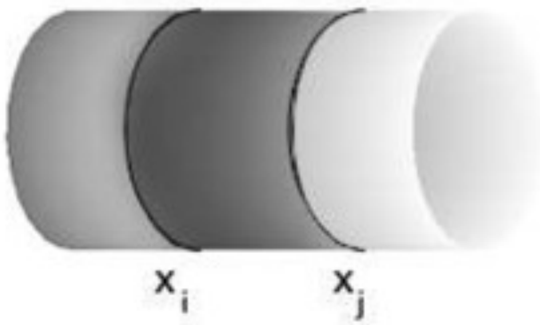
The BPS states come from cohomologies of the moduli of holomorphic curves embedded in the geometry, endowed with a flat $U(1)$ bundle, and with boundaries on the M5 branes.

In the present case, all the holomorphic curves are annuli, which are isolated.

The moduli of the bundle is an S^1

One gets precisely two cohomology classes,

corresponding to two BPS particles, for each pair of M5 branes.



From the M-theory perspective, the simplicity of the spectrum

is the key to solvability of the theory.

From this one can deduce the refined Chern-Simons partition function on M_L as a function of holonomies on the boundary

$$\langle x_I | 0 \rangle = \prod_{m=0}^{\beta-1} \prod_{1 \le I < J \le N} \left(q^{-m/2} e^{(x_J - x_I)/2} - q^{m/2} e^{(x_I - x_J)/2}\right) = \Delta_{q,t}(x)$$

By gluing, and adding more branes, one can compute the refined Chern-Simons amplitudes corresponding to S and T matrices

For example, the partition function on M_L with an unknot in representation R_i corresponds to adding one more stack of branes to $T^* M_L$

$$\langle x_I | R_i \rangle = \Delta_{q,t}(x) M_{R_i}(e^x)$$

Taking two copies of this, and gluing with a twist, we obtain $T^{*}S^{3}$ with with branes, whose partition function is computing the TST matrix element of the refined Chern-Simons theory

$$(TST)_{ij} = \int d^N x \, \Delta^2_{q,t}(x) \, e^{-\frac{1}{2\epsilon}\mathrm{Tr}\, x^2} M_{R_i}(e^{-x}) M_{R_j}(e^{x})$$

The fact that this takes the explicit form we gave before

$$S_{\bar{i}j}/S_{00} = M_{R_i}(t^{\rho}) M_{R_j}(q^{\lambda_{R_i}} t^{\rho})$$

$$T_{\bar{i}j} = T_i \delta^i_j = c_T q^{\frac{1}{2}(\lambda_{R_i}+\beta\rho)^2} \delta^i_j$$

was proven in the '90's in the context of proving Macdonald Conjectures

Cherednik, Kirillov, Etingof

The Verlinde coefficients N_{ijk}

compute the amplitude on $S^2 \times S^1$

with there Wilson lines wrapping the S^1

$$Z(S^2 \times S^1, R_i, R_j, R_k) = \frac{1}{N!} \int d^N x \, \Delta_{q,t}(x)^2 \, M_{R_i}(e^x) \, M_{R_j}(e^x) \, M_{R_k}(e^x)$$

$$= N_{ijk}$$

They correspond to three stacks of branes in $T^*\left(S^2 \times S^1\right)$

in addition to the N branes on $S^2 \times S^1$

They satisfy the Verlinde formula,

$$N_{ij\bar{k}} = \sum_{\ell} S_{\bar{\ell}i}\, S_{\bar{\ell}j}\, (S^*)_{\bar{k}}{}^{\bar{\ell}} / S_{\bar{\ell}0}$$

as required by three-dimensional topological invariance.

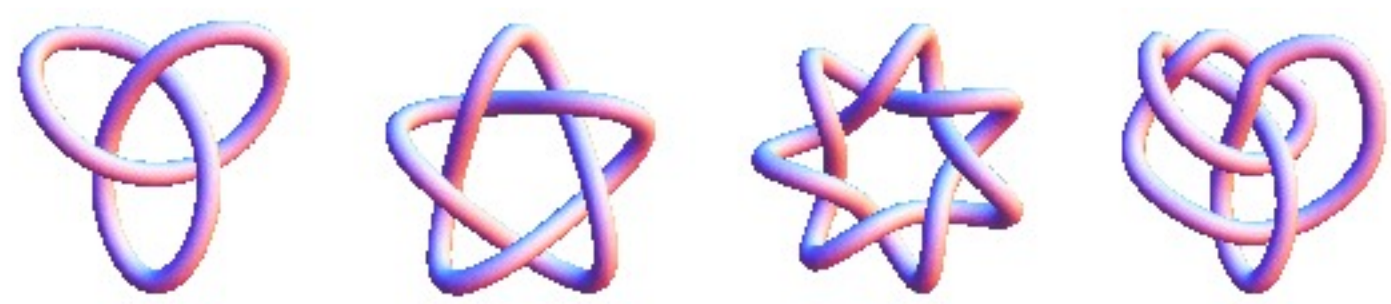
From S and T, we can obtain invariants of Seifert three manifolds, and knots in them which preserve the $U(1)$ action.

The case of most interest for knot theory is

$$M = S^3$$

where this corresponds to torus knots and links.

For example, the invariant of an (n,m) torus knot in S^3



(3,2) (5,2) (7,2) (4,3)

is computed by

$$Z(S^3, K, R_i) = \sum_{j,k,\ell} K_{0k} N^k{}_{ij} (K^{-1})^j{}_\ell S^\ell{}_p$$

where K represents an element of $SL(2,\mathbb{Z})$ that takes

the $(0,1)$ cycle into (n,m) cycle

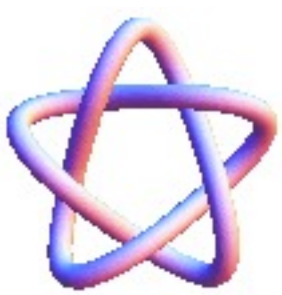
For knots colored with fundamental representation of $SU(N)$,

the normalized expectation value

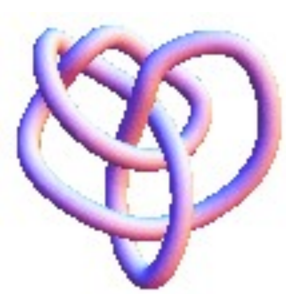
$$\mathcal{P}_{R_i}(K) = Z(S^3, K, R_i)/Z(S^3, \bigcirc, R_i)$$

equals

$$t^{-1/2+N}q^{-1/2} + t^{1/2+N}q^{1/2} - t^{-1/2+2N}q^{1/2}$$



$$t^{2N-1}q^{-1} + t^{2N} + t^{2N+1}q - t^{3N-1} - t^{3N}q$$



$$t^{3N-1/2}q^{1/2} + t^{3N+3/2}q^{1/2} + t^{3N+1/2}q^{1/2} + t^{3N-1/2}q^{-1/2}$$
$$+t^{3N-3/2}q^{5/2} - t^{4N+1/2}q^{3/2} - t^{4N-1/2}q^{3/2} - t^{4N-1/2}q^{1/2}$$
$$-t^{4N-3/2}q^{1/2} - t^{4N-3/2}q^{-1/2} + t^{5N-3/2}q^{3/2}$$

The knot invariants that arise are related to Poincare polynomials of the knot homology theory categorifying the HOMFLY polynomial.

We conjecture that the invariant of the knot

$$\mathcal{P}_{R_i}(K) = Z(S^3, K, R_i)/Z(S^3, \bigcirc, R_i)$$

with N dependence absorbed by

$$\lambda = t^N t^{1/2} q^{-1/2}$$

computes the Poincare polynomial

$$\mathcal{P}_{R_i}(K) = \mathcal{P}_{R_i}(K)(\mathbf{q}, \mathbf{a}, \mathbf{t})$$

of the reduced knot homology theory categorifying

the colored HOMFLY polynomial,

when written in terms of $\mathbf{q} = \sqrt{t},\ \mathbf{t} = -\sqrt{q/t}, \quad \mathbf{a} = \sqrt{\lambda},$

We have shown the conjecture holds for $(2,2m+1)$ knots,

for all m,

and $(3,4)$,$(3,5)$,$(3,7)$, and $(3,8)$ knots, colored by fundamental representation,

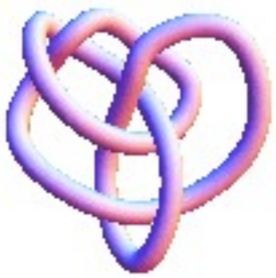
as our results reproduce the work of Rasmussen ’05, ’06

Changing variables to $\lambda = t^N t^{1/2} q^{-1/2}$ and

$$\mathbf{q} = \sqrt{t}, \qquad \mathbf{t} = -\sqrt{q/t}, \qquad \mathbf{a} = \sqrt{\lambda},$$

$$\mathcal{P}(K_{2,3})(\mathbf{q},\mathbf{a},\mathbf{t}) = \mathbf{a}^2\mathbf{q}^{-2} + \mathbf{a}^2\mathbf{q}^2\mathbf{t}^2 + \mathbf{a}^4\mathbf{t}^3$$

$$\mathcal{P}(K_{2,5})\,(\mathbf{q},\mathbf{a},\mathbf{t}) = \mathbf{a}^4\mathbf{q}^{-4} + \mathbf{a}^4\mathbf{t}^2 + \mathbf{a}^4\mathbf{q}^4\mathbf{t}^4 + \mathbf{a}^6\mathbf{q}^{-2}\mathbf{t}^3 + \mathbf{a}^6\mathbf{q}^2\mathbf{t}^5$$



$$\mathcal{P}(K_{3,4})\,(\mathbf{q},\mathbf{a},\mathbf{t}) = \mathbf{a}^6\mathbf{t}^4 + \mathbf{a}^6\mathbf{q}^6\mathbf{t}^6 + \mathbf{a}^6\mathbf{q}^2\mathbf{t}^4 + \mathbf{a}^6\mathbf{q}^{-2}\mathbf{t}^2 + \mathbf{a}^6\mathbf{q}^{-6} +$$
$$+\mathbf{a}^8\mathbf{q}^4\mathbf{t}^7 + \mathbf{a}^8\mathbf{q}^2\mathbf{t}^7 + \mathbf{a}^8\mathbf{t}^5 + \mathbf{a}^8\mathbf{q}^{-2}\mathbf{t}^5 + \mathbf{a}^8\mathbf{q}^{-4}\mathbf{t}^3 + \mathbf{a}^{10}\mathbf{t}^8$$

There is a geometric interpretation for the change of variables

$$\lambda = t^N t^{1/2} q^{-1/2}$$

in terms of a large N duality.

The Large N duality also provides a physical way to understand why the knot invariants are polynomials with integer coefficients.

Ooguri-Vafa, Marino-Labastida-Vafa

Gukov-Vafa-Schwarz

This fact is not manifest in the refined Chern-Simons theory, just like it is not manifest in the ordinary Chern-Simons theory.

The $SU(N)$ Chern-Simons theory on S^3,

or equivalently, the ordinary open topological string on

$$Y = T^*S^3$$

with N D-branes on S^3 is

dual to closed topological string on

$$X = O(-1) \oplus O(-1) \to P^1$$

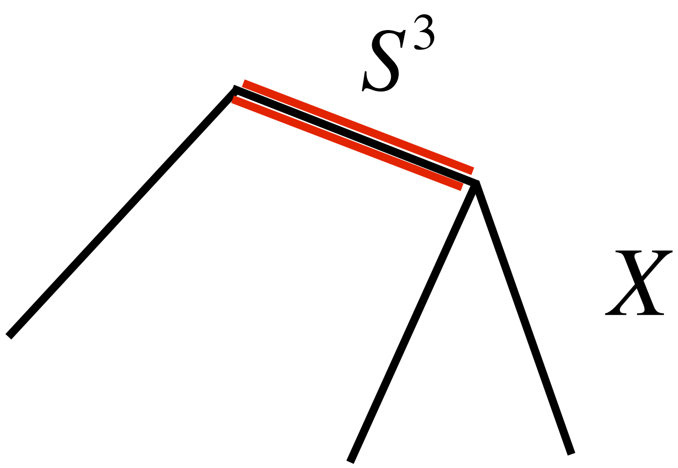
The duality is a geometric transition that

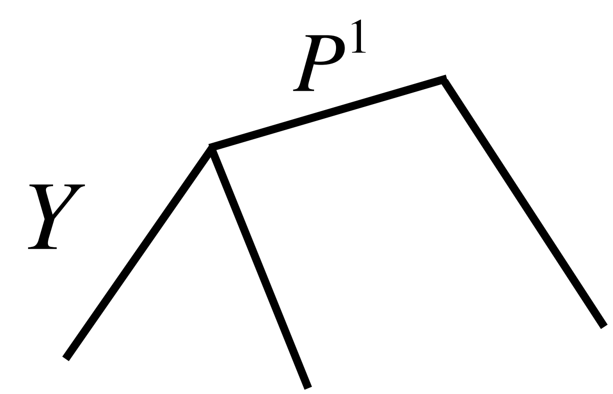
shrinks the S^3 and grows the P^1

Gopakumar-Vafa

of size $\lambda = e^{Area(P^1)}$ where

$$\lambda = q^N$$





Adding a knot K on S^3

Ooguri-Vafa

corresponds to adding a Lagrangian L_K in Y,

and geometric transition relates this to a Lagrangian L_K in X.

Ooguri and Vafa explained in '99 that the large N duality makes integrality of the knot polynomials manifest, provided one interprets the topological string on X in terms of M-theory on

$$(X \times TN \times S^1)_q$$

with M5 branes on

$$(L_K \times \mathbb{C} \times S^1)_q$$

The knot invariants are computed by the M-theory index,

$$Z_{M5}(M,q) = Tr\left(-1\right)^F q^{S_1 - S_2}$$

counting now M2 branes ending on the M5 branes on $L_K \in X$

While integrality of knot polynomials is not manifest in Chern-Simons theory on S^3,

the relation to counting of BPS states in M-theory on

$$X = O(-1) \oplus O(-1) \to P^1$$

makes the integrality manifest.

The large N duality extends to the refined Chern-Simons theory and the refined topological string.

The partition function of
the refined SU(N) Chern-Simons theory on S^3

$$Z(S^3, q, t, N) = Z^{top}(X, q, t, \lambda)$$

equals to partition function of the refined topological string on

$$X = O(-1) \oplus O(-1) \rightarrow P^1$$

with

$$\lambda = t^N t^{\frac{1}{2}} q^{-\frac{1}{2}}$$

The large N duality now relates computing knot invariants of the refined Chern-Simons theory on S^3

to computing the refined index

$$Z_{M5}(L_K, X, q, t) = Tr\left(-1\right)^F q^{S_1 - S_R} t^{S_R - S_2}$$

counting M2 branes ending on M5 branes wrapping

$$(L_K \times \mathbb{C} \times S^1)_{q,t}$$

in M-theory on

$$(X \times TN \times S^1)_{q,t}$$

Thus, rewriting the knot invariants of the refined Chern-Simons theory

$$\mathcal{P}_{R_i}(K) = Z(S^3, K, R_i)/Z(S^3, \bigcirc, R_i)$$

in terms of

$$\lambda = t^N t^{1/2} q^{-1/2}$$

corresponds to using variables natural in the large N dual.

Our results provide support for the conjecture

of Gukov, Vafa and Schwarz, that

the knot homologies

$$H^{i,j,k}$$

of a triply graded homology theory categorifying the HOMFLY polynomial

are, up to regrading, the spaces of BPS states in M-theory on

$$X = O(-1) \oplus O(-1) \to P^1$$

graded by the spins S_1, S_2 and their charge in $H^2(X,\mathbb{Z})$.

This conjecture (which in part motivated this work) is more general than the present results, as it does not require existence of a the $U(1)_R$ symmetry, needed to formulate the partition function of the refined open topological string.

At the same time, in this more general setting, one looses the power of a topological field theory in three dimensions.