Topological T-duality With Monodromy

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- Monodromy around singular fibers

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We would like to bridge the gap between these constructions

Top T-duality can incorporate monodromy, singularities not yet considered

Affine Torus Bundles

Let $V \simeq \mathbb{R}^n, \quad \Lambda \simeq \mathbb{Z}^n, \quad T^n = V/\Lambda$

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$\mathrm{Aff}(T^n) \to \mathrm{Diff}(T^n)$ is a homotopy equivalence for $n \leq 3$.

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$(\rho, c) \simeq (\rho', c')$ if they are related by an element of $\mathrm{GL}(n, \mathbb{Z})$

Topological T-duality (rank 1 case)

$$\pi : E \to M, \quad \hat{\pi} : \hat{E} \to M \quad \text{circle bundles on } M$$

$$h \in H^3(E, \mathbb{Z}), \quad \hat{h} \in H^3(\hat{E}, \mathbb{Z}) \quad \text{flux on } E, \hat{E}$$

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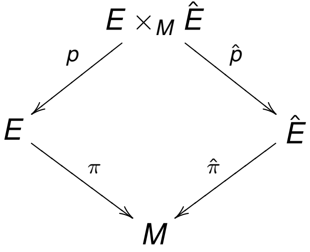

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$(E, h), (\hat{E}, \hat{h})$ are **T-dual** if

- $\rho = \hat{\rho}$ (dual monodromy)
- $\pi_*(h) = c_1(\hat{E}) \in H^2(M, \mathbb{Z}_{\hat{\rho}})$ (swap Chern class and flux)
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Theorem

For any (E, h) there exists a T-dual $(\hat{E}, \hat{h})$ unique up to isomorphism (fibre bundle isomorphisms)

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Leray-Serre spec seq for $\pi : E \to M$ yields filtration

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we demand $h \in F^{2,3}(\pi)$ (h has "one leg on the fiber")

$F^{2,3}(\pi) \ni h \mapsto [h] \in F^{2,3}(\pi)/F^{3,3}(\pi) = E_\infty^{2,1}(\pi)$

$E_\infty^{2,1}(\pi)$ is a subquotient of $E_2^{2,1}(\pi) = H^2(M, \Lambda_\rho^*)$

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$(E, h), (\hat{E}, \hat{h})$ are **T-dual** if

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Theorem

For any (E, h) there exists a T-dual $(\hat{E}, \hat{h})$. The Chern class of $\hat{E}$ is determined up to a map $H^0(M, \wedge^2 \Lambda_\rho^*) \to H^2(M, \Lambda_\rho^*)$ given by contraction with $c_1(E)$.

Main Step in Proof

Can find $(\hat{E}, h')$ where $c_1(\hat{E})$ represents $[h]$ and $c_1(E)$ represents $[h']$

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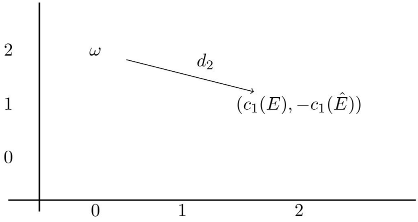

$p^* h - \hat{p}^* h'$ is represented by $(c_1(E), -c_1(\hat{E}))$

so $p^* h - \hat{p}^* h' = \hat{p}^* \hat{\pi}^*(a)$

we then set $\hat{h} = h' + \hat{\pi}^* a$

Twisted Cohomology

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Assume $E, \hat{E}$ smooth, $H, \hat{H}$ closed 3-forms representing $h, \hat{h}$.

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Theorem

If ρ is $\mathrm{SL}(n, \mathbb{Z})$-valued then we have an isomorphism

$$T : H^k(E, H) \simeq H^{k-n}(\hat{E}, \hat{H})$$

$$T\alpha = \int_{\hat{T}^n} e^{\mathcal{B}} \pi^*(\alpha)$$

where $\mathcal{B}$ is a certain 2-form on $E \times_M \hat{E}$ which restricted to the fibers $T^n \times \hat{T}^n$ is the natural symplectic form ω

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We have isomorphisms

$$\begin{aligned} H^k(E, H) &\simeq H^{k-n}(\hat{E}, (w_1, \hat{H})) \\ H^k(E, (w_1, H)) &\simeq H^{k-n}(\hat{E}, \hat{H}) \end{aligned}$$

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We have isomorphisms

$$\begin{aligned} K^k(E, h) &\simeq K^{k-n}(\hat{E}, (w_1, \hat{h} + W_3)) \\ K^k(E, (w_1, h + W_3)) &\simeq K^{k-n}(\hat{E}, \hat{h}) \end{aligned}$$

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In the special case $(w_1, W_3) = (0, 0)$ this reduces to

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Modulo $p^*H^2(T_m, \mathbb{Z}) + \hat{p}^*H^2(\hat{T}_m, \mathbb{Z})$ required to be the Poincaré line bundle

Proof (following Bunke Rumpf Schick)

Fourier-Mukai type transformation

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Locally T looks like K-theoretic Fourier-Mukai

use Mayer-Vietoris

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THANK YOU