"RATIONAL" MATRIX FACTORIZATIONS AND DEFECTS VIA CLOSED FUNCTOR ALGEBRA

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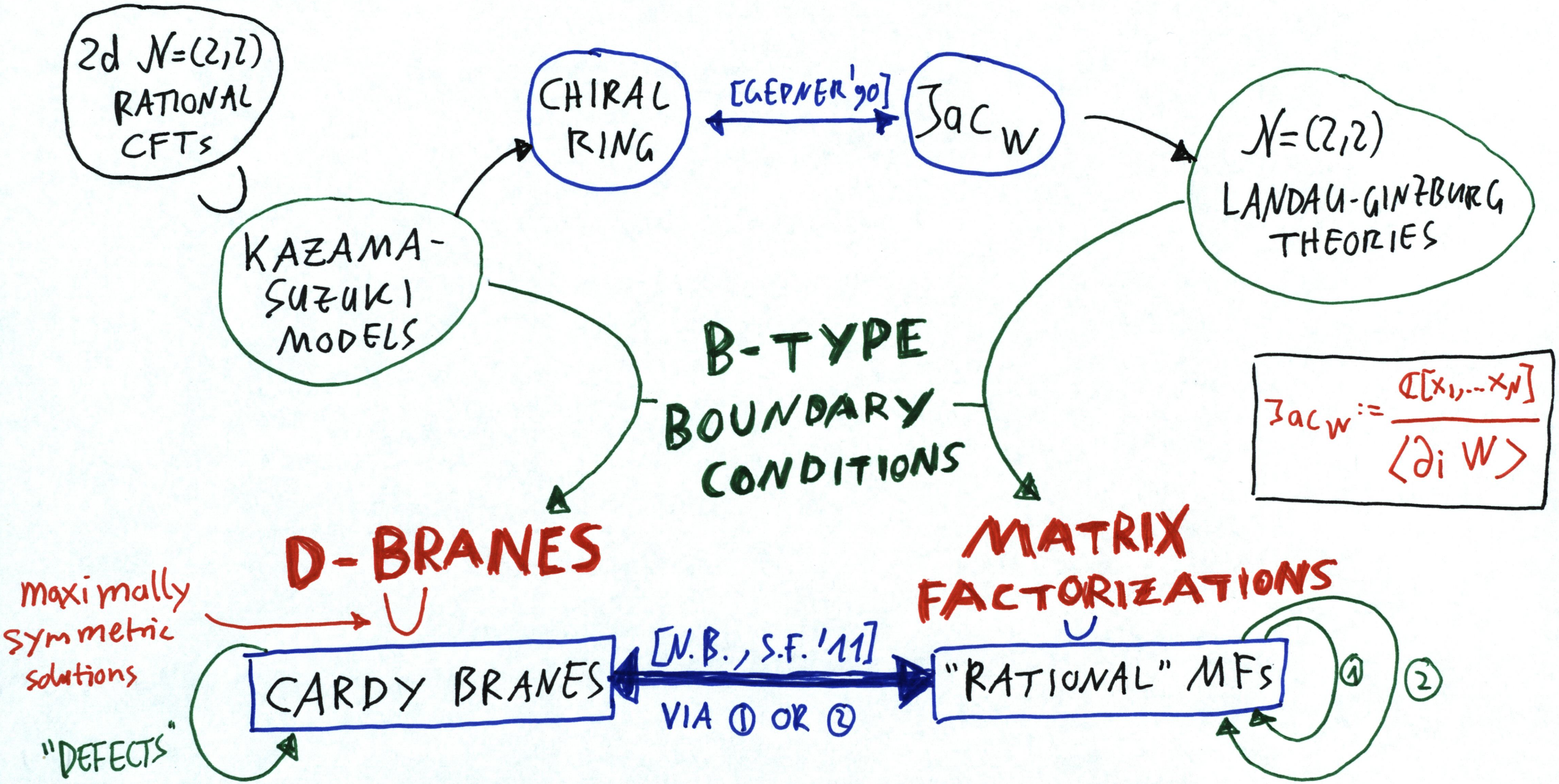
BASED ON: [1005.2117]
[110X.XXXX]
(W/ STEFAN FREDENHAGEN)

MOTIVATION: LANDAU-GINZBURG / RCFT CORRESPONDENCE

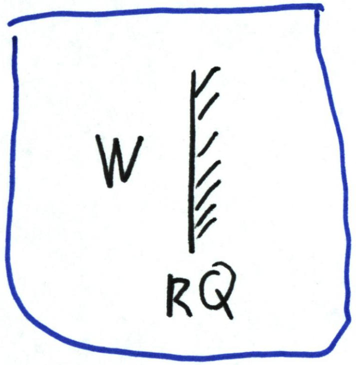


RCFT: { DEFECTS } ACTING ON { "ELEMENTARY" CARDY BRANES } ⟹ { ALL CARDY BRANES }

LANDAU-GINZBURG THEORIES

FIX SOME RING R (HERE: $R = \mathbb{C}[x_1, \dots, x_N]$), $W \in R$ ("SUPERPOTENTIAL")

GEPNER!



- SOLUTIONS TO B-TYPE BOUNDARY CONDITIONS:

DEF.: MATRIX FACTORIZATION $_RQ$ OF TYPE (R,W)

$$\left({}_RM^0 \xrightarrow{{}_Rp^0} {}_RM^1 \xrightarrow{{}_Rp^1} {}_RM^0 \right)$$

- $_RM^i$ – FREE R-MODULES
- $_Rp^0 \circ {}_Rp^1 = W \cdot \mathrm{id}_{M^0} \wedge {}_Rp^1 \circ {}_Rp^0 = W \cdot \mathrm{id}_{M^1}$

↳ SHORTHAND NOTATION: $\boxed{{}_RQ := \begin{pmatrix} 0 & {}_Rp^1 \\ {}_Rp^0 & 0 \end{pmatrix}}$

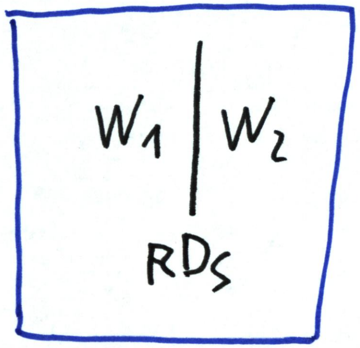
- LET $\phi : R \to S$ RING HOMOMORPHISM ↷ DEF.: $\boxed{\phi^* : R\text{-mod} \rightleftarrows S\text{-mod} : \phi_*}$
  - $\phi^* := - \otimes_R S$
  - ϕ_* VIA $\underset{R}{\underset{\cap}{r}} . \underset{{}_SM}{\underset{\cap}{m}} := \phi(r) . m$

IDEA ①: ACT w/ ϕ^*, ϕ_* ON MFs!

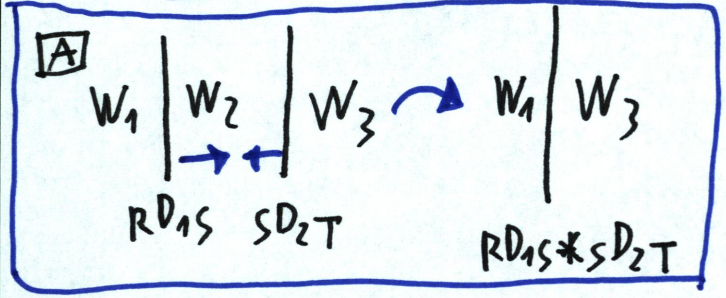
DEFECTS IN LANDAU-GINZBURG THEORIES

■ **BASIC DEFINITION:** DEFECT-MF OF TYPE (R, S, W_1, W_2), $W_1 \in R$, $W_2 \in S$

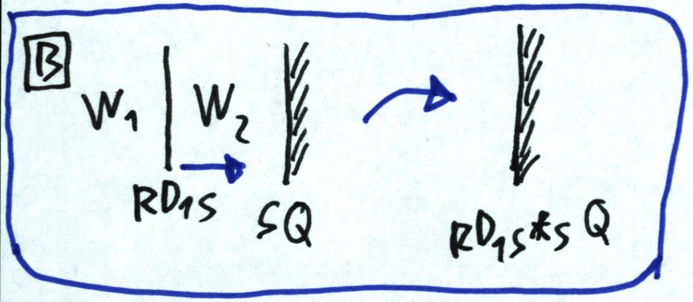
$$_R D_S := \text{"MF OF } W_1 \otimes 1_S - 1_R \otimes W_2\text{"}$$



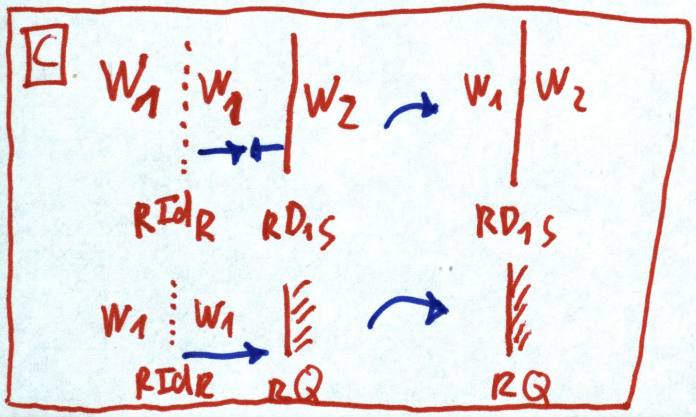
■ **FUSION** (OF DEFECTS/DEFECTS OR DEFECTS/MFs)



A $\;{}_R D_{1S} *_S {}_S D_{2T} := \left({}_R D_{1S} \hat{\otimes}_S {}_S D_{2T}\right)\Big|_{\text{"FORGET" } S}$ = DEFECT MF OF TYPE (R, T, W_1, W_3)



B $\;{}_R D_{1S} *_S Q := \left({}_R D_{1S} \hat{\otimes}_S Q\right)\Big|_{\text{"FORGET" } S}$ = MF OF TYPE (R, W_1)



C SIMPLEST DEFECT MFS: "IDENTITY" DEFECT MFs

$$_R Id_R *_R {}_R D_{1S} = {}_R D_{1S} \;;\quad {}_R Id_R *_R {}_R Q = {}_R Q \quad \forall\, {}_R D_{1S}, {}_R Q$$

IDEA ②: LET $\phi : R \to S$ RING HOM., (R, W_1), (S, W_2) LG THEORIES

$$\hookrightarrow \; {}_R D^{\phi}_S := (\phi_*, 1)\, {}_S Id_S \;;\quad {}_S \tilde{D}^{\phi}_R := (\phi^*, 1)\, {}_R Id_R$$

III EXPLICIT EXAMPLE:

[1005.2117]

$SU(3)_k/U(2)$ KAZAMA-SUZUKI MODELS $\longleftrightarrow$ [GEPNER '90] LG THEORY $(\mathbb{C}[y_1, y_2], W_k)$

$$W_k(y_1, y_2) := \left(x_1^{k+3} + x_2^{k+3}\right)\Big|_{\substack{x_1+x_2 \to y_1 \\ x_1 \cdot x_2 \to y_2}}$$

- STARTING POINT FOR <u>BOUNDARY</u> LG/RCFT CORRESPONDENCE:

ANALYZE SIMPLEST ("<u>POLYNOMIAL</u>") MFS $\Rightarrow$

$$Q_{|1,0\rangle} := \begin{pmatrix} 0 & \mathfrak{z}_{|1,0\rangle} \\ \frac{W_k}{\mathfrak{z}_{|1,0\rangle}} & 0 \end{pmatrix} \longleftrightarrow \text{"}|1,0\rangle\text{"}$$

(POLYNOMIAL $\in \mathbb{C}[y_1, y_2]$; "ELEMENTARY" CARDY BRANES)

- STRATEGY FOR <u>FULL</u> "DICTIONARY":

[110x.xxxx]

$$\iota : \underbrace{\mathbb{C}[y_1, y_2]}_{=:R} \longrightarrow \underbrace{\mathbb{C}[x_1, x_2]}_{=:S} : \begin{cases} y_1 \mapsto x_1 + x_2 \\ y_2 \mapsto x_1 \cdot x_2 \end{cases} \quad \left(\text{i.e. } W_k \mapsto \widetilde{W}_k := x_1^{k+3} + x_2^{k+3}\right)$$

$$\sigma_k^{(m,n)} : S \longrightarrow S : \begin{cases} x_1 \mapsto x_1 \cdot \eta^m \\ x_2 \mapsto x_2 \cdot \eta^n \end{cases} \quad \left(\eta := \exp\left(\tfrac{2\pi i}{k+3}\right);\ \widetilde{W}_k \mapsto \widetilde{W}_k\right)$$

↳ <u>DEF.</u>:

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{D_k^{(m,n)}} & R\text{-mod} \\ \downarrow \iota^* & & \uparrow \iota_* \\ S\text{-mod} & \xrightarrow{(\sigma_k^{(m,n)})^*} & S\text{-mod} \end{array}$$

$$D_k^{(m,n)} := \iota_* \circ \left(\sigma_k^{(m,n)}\right)^* \circ \iota^*$$

$m, n \in \mathbb{Z}$

$k \in \mathbb{Z}_{>0}$

⇒ COMPLETE "BOUNDARY LG/RCFT DICTIONARY"

$SU(3)_k/U(2)$ KAZAMA-SUZUKI MODELS ⟷ [GEPNER '90] ⟷ **LG THEORIES $(\mathbb{C}[y_1,y_2], W_k)$**

$$W_k(y_1,y_2) := \left(x_1^{k+3} + x_2^{k+3}\right)\Big|_{\substack{x_1+x_2 \to y_1 \\ x_1 \cdot x_2 \to y_2}}$$

B-TYPE BOUNDARY CONDITIONS

$0 \leq L \leq \lfloor \frac{k}{2} \rfloor$

$0 \leq \ell \leq k+1$

CARDY BRANES "$|L,\ell\rangle$" ⟷ [NB, SF '11] ⟷ **"RATIONAL" MFs** $Q_{|L,\ell\rangle}$

"DEFECTS" (acting on Cardy branes); $\{D_{(\ell)}\}$ (acting on rational MFs)

RCFT: $\{$DEFECTS$\}$ ACTING ON $\{$"ELEMENTARY" CARDY BRANES$\}$ ("$|L,0\rangle$") $\Longrightarrow$ $\{$ALL CARDY BRANES$\}$ ("$|L,\ell\rangle$")

LG:

FUNCTOR ALGEBRA

$D_{(1)} := D_k^{(0,1)}$; $D_{(k+2)} \cong 0$

$D_{(1)} D_{(\ell)} = D_{(\ell-1)} \oplus D_{(\ell+1)}$

"POLYNOMIAL" MFs

$$Q_{|L,0\rangle} := \begin{pmatrix} 0 & \mathfrak{I}_{|L,0\rangle} \\ \frac{W_k}{\mathfrak{I}_{|L,0\rangle}} & 0 \end{pmatrix}$$

"RATIONAL" MFs

$$Q_{|L,\ell\rangle} := D_{(\ell)}\, Q_{|L,0\rangle}$$

"BONUS": DEFECT MFs VIA ϕ^*, ϕ_* - TYPE FUNCTORS

RELATION TO MATH LITERATURE: KHOVANOV-ROZANSKY KNOT HOMOLOGY

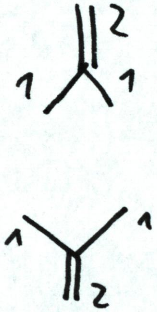
$R := \mathbb{C}[y_1, y_2]$

$\downarrow \iota$

$S := \mathbb{C}[x_1, x_2]$

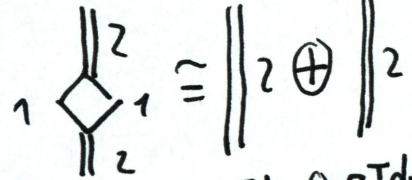
$${}_R X_S := (\iota_*, 1)\, {}_S\mathrm{Id}_S \;\hat{=}$$

$${}_S X_R := (\iota^*, 1)\, {}_R\mathrm{Id}_R \;\hat{=}$$



BASIC MOY GRAPHS

→ e.g.



$${}_R X_S * {}_S X_R \cong {}_R\mathrm{Id}_R \oplus {}_R\mathrm{Id}_R$$

RELATION TO PHYSICS LITERATURE:

$\mathbb{C}[x_1, x_2]$

$\downarrow \sigma_k^{(m,n)}$

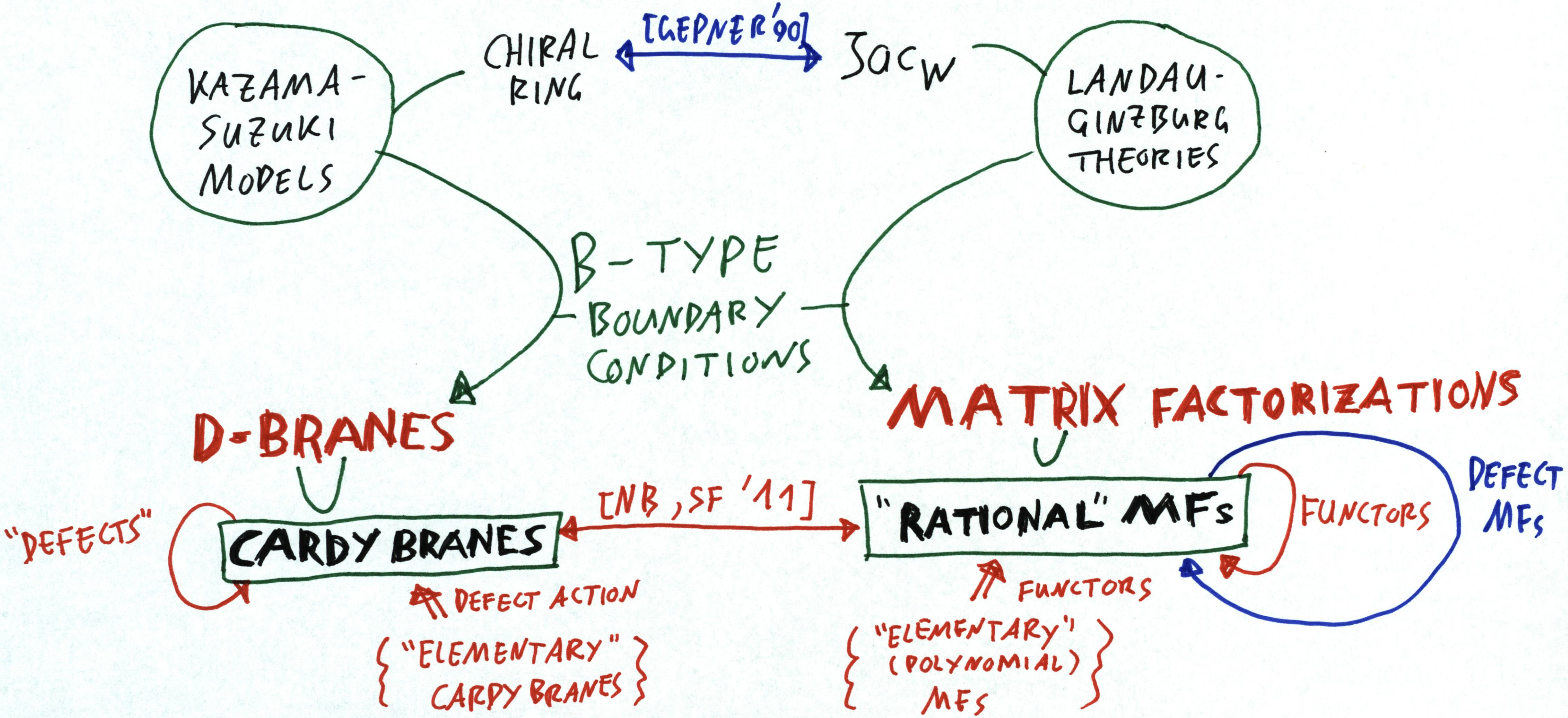
$\mathbb{C}[x_1, x_2]$

$${}_S \mathcal{G}^{(m,n)}_{k\,S} := \left(\left(\sigma_k^{(m,n)}\right)^*, 1\right) {}_S\mathrm{Id}_S$$

$${}_S \mathcal{G}^{(m_1,n_1)}_{k\,S} * {}_S \mathcal{G}^{(m_2,n_2)}_{k\,S} = {}_S \mathcal{G}^{(m_1+n_1,\, m_2+n_2)}_{k\,S}$$

"GROUP-LIKE" DEFECTS ↗

CONCLUSION



OUTLOOK:

- STUDY MORE GENERAL D-BRANES VIA THE CORRESPONDENCE
- ANALYZE BOUNDARY RG FLOWS (✓ FOR $SU(3)_k/U(2)$ MODELS!)
- LINK TO KHOVANOV-ROZANSKY LINK/KNOT HOMOLOGY?
- ESTABLISH "DICTIONARY" FOR OTHER RCFTS