Courant Algebroids and Generalizations of Geometry

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- $TM \oplus T^*M \oplus \mathfrak{G}$ (type I + YM, heterotic)
- $TM \oplus \wedge^2 T^*M \oplus \ldots$ (M-theory)

Outline of the talk

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- T-duality and generalized geometry

Generalized geometry [Hitchin, Gualtieri, Cavalcanti]

Replace structures on TM (such as $[\ ,\ ]$, $\imath_X$, $\mathcal{L}_X$, d, ...) by similar structures on $E = TM \oplus T^*M$

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$$\langle x + \xi, y + \eta \rangle = \frac{1}{2}(\imath_x \eta + \imath_y \xi)$$

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$$(x + \xi) \circ (y + \eta) = [x, y] + \mathcal{L}_x \eta - \imath_y d\xi$$

- Clifford algebra

$$\{\gamma_{x+\xi}, \gamma_{y+\eta}\} = 2\langle x + \xi, y + \eta \rangle$$

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Generalized Geometry (cont'd)

- Clifford module $\Omega^{\bullet}(M)$

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- De-Rham differential on $\Omega^\bullet(M)$

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

Symmetries

Symmetries of $\langle\ ,\ \rangle$ given by sections of the adjoint bundle

$$\wedge^2 E \cong \wedge^2 TM \oplus \mathrm{End}(TM) \oplus \wedge^2 T^*M$$

[They form the group $O(n,n)$]

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In particular, we have the so called B-transform, for $b \in \Omega^2(M)$

$$e^b \cdot (x+\xi) = x + (\xi + \imath_x b)$$
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Symmetries of the Dorfman bracket are $\mathrm{Diff}(M) \ltimes \Omega^2_{\mathrm{cl}}(M)$

Twisting

This suggest the introduction of a twisted Dorfman bracket, with $H \in \Omega^3(M)$, $dH = 0$

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and a twisted differential

$$d_H \omega = d\omega + H \wedge \omega$$

Properties (for $A, B, C \in \Gamma E$, $f \in C^\infty(M)$)

(i) $A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)$

(ii) $A \circ (fB) = f(A \circ B) + (\rho(A)f)B$

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The Courant bracket is defined as the anti-symmetrization

$$[\![A, B]\!] = \frac{1}{2}(A \circ B - B \circ A)$$

or, conversely,

$$A \circ B = [\![A, B]\!] + d\langle A, B\rangle$$

Dorfman bracket as a derived bracket

Recall the usual Cartan relations

$$\begin{aligned}
\{\imath_x, \imath_y\} &= 0 \\
\{d, \imath_x\} &= \mathcal{L}_x \\
[\mathcal{L}_x, \imath_y] &= \imath_{[x,y]} \\
[\mathcal{L}_x, \mathcal{L}_y] &= \mathcal{L}_{[x,y]} \\
[d, \mathcal{L}_x] &= 0
\end{aligned}$$

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[d, \mathcal{L}_x] &= 0
\end{aligned}
$$

Here we have the analogue

$$
\begin{aligned}
\{\gamma_A, \gamma_B\} &= 2\langle A, B\rangle \\
\{d_H, \gamma_A\} &= \mathcal{L}_A \\
[\mathcal{L}_A, \gamma_B] &= \gamma_{A\circ B} \\
[\mathcal{L}_A, \mathcal{L}_B] &= \mathcal{L}_{A\circ B} = \mathcal{L}_{[\![A,B]\!]} \\
[d_H, \mathcal{L}_A] &= 0
\end{aligned}
$$

where $\mathcal{L}_{x+\xi}\omega = \mathcal{L}_x\omega + (d\xi + \imath_x H) \wedge \omega$

Leibniz algebroids

Definition

A Leibniz algebroid $(E, \circ, \rho)$ is a vector bundle $E \to M$, with a composition (Leibniz/Loday bracket) $\circ$ on ΓE, and a morphism of vector bundles $\rho : E \to TM$ (anchor) such that

(L1) $A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)$

(L2) $\rho(A \circ B) = [\rho(A), \rho(B)]$

(L3) $A \circ (fB) = f(A \circ B) + (\rho(A)f)B$

Courant algebroids

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(C1) $A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)$

(C2) $\rho(A)\langle B, C\rangle = \langle A, B \circ C + C \circ B\rangle$

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It follows that $(E, \circ, \rho)$ is a Leibniz algebroid. Moreover

$$A \circ B + B \circ A = 2D\langle A, B\rangle$$

where $D = \frac{1}{2}\rho^* d : C^\infty(M) \to \Gamma E$, i.e. $\langle Df, A\rangle = \frac{1}{2}\rho(A)f$

Exact Courant algebroids

An exact Courant algebroid E is a Courant algebroid that fits in the exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0$$

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Such a Courant algebroid admits an isotropic splitting $s : TM \to E$, which allows us to identify $E \cong TM \oplus T^*M$. The composition on $x + \xi \in \Gamma(TM \oplus T^*M)$ is uniquely determined by

$$H(x, y, z) = \langle x \circ y, z \rangle$$

It turns out $H \in \Omega^3_{\mathrm{cl}}(M)$

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Theorem (Ševera)

Equivalence classes of exact Courant algebroids are in 1–1 correspondence with $H^3(M, \mathbb{R})$.

B_{2n}-geometry [Baraglia]

We now consider the vector bundle

$$E = TM \oplus 1 \oplus T^*M$$

with nondegenerate bilinear form

$$\langle x + f + \xi, y + g + \eta \rangle = \frac{1}{2}(\imath_x \eta + \imath_y \xi) + fg$$

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$$(x+f+\xi)\circ(y+g+\eta) = [x,y] + (x(g) - y(f)) + \mathcal{L}_x \eta - \imath_y d\xi + 2g df$$

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and anchor map

$$\rho(x + f + \xi) = x$$

Adjoint bundle

$$\wedge^2 E = \wedge^2 TM \oplus TM \oplus \mathrm{End}(TM) \oplus T^*M \oplus \wedge^2 T^*M$$

B_{2n}-geometry (cont'd)

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In particular

$$a \cdot (x + f + \xi) = \imath_x a - fa \,, \qquad a \in \Omega^1$$
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Note

$$[a_1, a_2] = a_1 \wedge a_2$$

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Symmetries of the Dorfman bracket iff $da = 0 = db$.

B_{2n}-geometry (cont'd)

Consider $F \in \Omega^2(M)$, $H \in \Omega^3(M)$, and a twisted Dorfman bracket

$$
\begin{aligned}
(x + f + \xi) \circ (y + g + \eta) = [x, y] + (x(g) - y(f)) + \imath_x \imath_y F \\
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This defines a Courant algebroid provided

$$dF = 0$$
$$dH + F \wedge F = 0$$

B_{2n}-geometry (cont'd)

The bracket can be obtained as a derived bracket using

$$d_{F,H}\,\omega = d\omega + (-1)^{|\omega|} F \wedge \omega + H \wedge \omega$$

and

$$\gamma_{x+f+\xi} \cdot \omega = \imath_x \omega + (-1)^{|\omega|} f\omega + \xi \wedge \omega$$

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Note

$$d_{F,H}{}^2 = 0$$
$$\left\{\gamma_{x+f+\xi}, \gamma_{y+g+\eta}\right\} = 2\langle x+f+\xi, y+g+\eta\rangle$$

iff F and H satisfy the Bianchi identities as before.

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iff F and H satisfy the Bianchi identities as before.
This is an example of non-exact transitive Courant algebroid

$$T^*M \longrightarrow TM \oplus 1 \oplus T^*M \longrightarrow TM \longrightarrow 0$$

Transitive Courant Algebroids [Chen-Stiénon-Xu]

Transitive Courant algebroids are of the form

$$E = TM \oplus \mathfrak{G} \oplus T^*M$$

where $\mathfrak{G} = \mathrm{ker}\rho/(\mathrm{ker}\rho)^{\perp}$ is a bundle of Lie algebras with bracket

$$[r, s]_{\mathfrak{G}} = \mathrm{pr}_{\mathfrak{G}}(r \circ s)$$

and

$$\rho(x + r + \xi) = x$$

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and

$$\rho(x + r + \xi) = x$$

Suppose

$$\langle x + r + \xi, y + s + \eta \rangle = \frac{1}{2}\left(\imath_x \eta + \imath_y \xi\right) + \langle r, s \rangle_{\mathfrak{G}}$$

Transitive Courant Algebroids (cont'd)

Then the Dorfman bracket on E is completely determined by

$$\begin{aligned} H(x,y,z) &= \langle \mathrm{pr}_{T^*M}(x \circ y), z \rangle \\ R(x,y) &= \mathrm{pr}_{\mathfrak{G}}(x \circ y) \\ \nabla_x r &= \mathrm{pr}_{\mathfrak{G}}(x \circ r)\,. \end{aligned}$$

It turns out $H \in \Omega^3(M)$, $R \in \Omega^2(M, \mathfrak{g})$, and ∇_x a TM-connection on $\mathfrak{G}$

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Namely

$$\begin{aligned}(x + r + \xi) \circ (y + s + \eta) = {} & [x,y] \\ & - \imath_x \imath_y R + [r,s]_{\mathfrak{G}} + \nabla_x s - \nabla_y r \\ & - \imath_x \imath_y H + \langle s, d_\nabla r\rangle_{\mathfrak{G}} + \mathcal{L}_x \eta - \imath_y d\xi + \langle \imath_x R, s\rangle_{\mathfrak{G}} - \langle \imath_y R, r\rangle_{\mathfrak{G}}\end{aligned}$$

where we have defined a 'twisted differential' $d_\nabla : \Omega^k(M, \Gamma\mathfrak{G}) \to \Omega^{k+1}(M, \Gamma\mathfrak{G})$ by

$$(d_\nabla\omega)(x_0, \dots x_k) = \sum_{i=0}^{k} (-1)^i \nabla_{x_i}\omega(x_0, \dots, \widehat{x_i}, \dots, x_k)$$
$$+ \sum_{0\le i<j\le k} (-1)^{i+j}\omega([x_i, x_j], x_0, \dots, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_k)$$

[Note that $d_\nabla^2 = 0$ iff the curvature corresponding to ∇_x vanishes]

Transitive Courant Algebroids (cont'd)

where

$$
\begin{aligned}
\mathcal{L}_x \langle r, s\rangle_{\mathfrak{G}} &= \langle \nabla_x r, s\rangle_{\mathfrak{G}} + \langle r, \nabla_x s\rangle_{\mathfrak{G}} \\
\nabla_x [r, s]_{\mathfrak{G}} &= [\nabla_x r, s]_{\mathfrak{G}} + [r, \nabla_x s]_{\mathfrak{G}} \\
d_\nabla R &= 0 \\
d_\nabla^2 r &= \left(\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}\right) r = [R, r]_{\mathfrak{G}} \,, \\
dH &= \langle R \wedge R\rangle_{\mathfrak{G}} \,.
\end{aligned}
$$

M-geometry [Hull, Pacheco-Waldram, Baraglia]

In M-theory we have a 3-form C_3 with

$$F_4 = dC_3$$

satisfying

$$dF_4 = 0 \qquad \text{(Bianchi)}$$

and

$$d(*F_4) + \tfrac{1}{2} F_4 \wedge F_4 = 0 \qquad \text{(e.o.m.)}$$

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After putting $F_7 = *F_4$ we have

$$d(F_7 + \tfrac{1}{2} C_3 \wedge F_4) = 0\,, \qquad F_7 + \tfrac{1}{2} C_3 \wedge F_4 = dC_6$$

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M-geometry (cont'd)

Summarizing

$$F_4 = dC_3$$
$$F_7 = dC_6 - \tfrac{1}{2} C_3 \wedge F_4$$

with

$$dF_4 = 0$$
$$dF_7 + \tfrac{1}{2} F_4 \wedge F_4 = 0$$

M-geometry (cont'd)

Symmetries by $z_3 \in \Omega^3_{\text{cl}}$, $z_6 \in \Omega^6_{\text{cl}}$

$$\begin{aligned} C'_3 &= C_3 + z_3 \\ C'_6 &= C_6 + z_6 + \tfrac{1}{2} C_3 \wedge z_3 \end{aligned}$$

Group law

$$(z_3, z_6) \cdot (z'_3, z'_6) = (z_3 + z'_3, z_6 + z'_6 - \tfrac{1}{2} z_3 \wedge z'_3)$$

M-geometry (cont'd)

The relevant bundle in this case is

$$E = TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$$

with Dorfman bracket

$$(x + a_2 + a_5) \circ (y + b_2 + b_5) = \\ [x, y] + \mathcal{L}_x b_2 - \imath_y da_2 + \mathcal{L}_x b_5 - \imath_y da_5 + da_2 \wedge b_2$$

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$$[x, y] + \mathcal{L}_x b_2 - \imath_y da_2 + \mathcal{L}_x b_5 - \imath_y da_5 + da_2 \wedge b_2$$

The bracket is invariant under infinitesimal symmetries generated by $z_3 \in \Omega^3_{\text{cl}}$, $z_6 \in \Omega^6_{\text{cl}}$

$$z_3 \cdot (x + a_2 + a_5) = \imath_x z_3 - z_3 \wedge a_2$$
$$z_6 \cdot (x + a_2 + a_5) = -\imath_x z_6$$

.

M-geometry (cont'd)

It can be twisted by $F_4 \in \Omega^4$, $F_7 \in \Omega^7$

$$
\begin{aligned}
(x + a_2 + a_5) \circ (y + b_2 + b_5) = {} & [x, y] \\
& + \mathcal{L}_x b_2 - \imath_y da_2 + \imath_x \imath_y F_4 \\
& + \mathcal{L}_x b_5 - \imath_y da_5 + da_2 \wedge b_2 + \imath_x F_4 \wedge b_2 + \imath_x \imath_y F_7
\end{aligned}
$$

M-geometry (cont'd)

It can be twisted by $F_4 \in \Omega^4$, $F_7 \in \Omega^7$

$$\begin{aligned}(x + a_2 + a_5) \circ (y + b_2 + b_5) = {} & [x, y] \\ & + \mathcal{L}_x b_2 - \imath_y da_2 + \imath_x \imath_y F_4 \\ & + \mathcal{L}_x b_5 - \imath_y da_5 + da_2 \wedge b_2 + \imath_x F_4 \wedge b_2 + \imath_x \imath_y F_7\end{aligned}$$

and is a Leibniz algebroid iff

$$\begin{aligned} dF_4 &= 0 \\ dF_7 + \tfrac{1}{2} F_4 \wedge F_4 &= 0 \end{aligned}$$

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Note that we have $\langle\ ,\ \rangle : E \otimes E \to T^*M \oplus \wedge^4 T^*M$

$$\langle x + a_2 + a_5, y + b_2 + b_5 \rangle = (\imath_x b_2 + \imath_y a_2) + (\imath_x b_5 + \imath_y a_5 + a_2 \wedge b_2)$$

(cf. notion of E-Courant algebroid [Chen-Liu-Sheng])

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Suppose we have a pair (E, H), consisting of a principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

and a so-called H-flux H, a Čech 3-cocycle.

Topologically, E is classified by an element in $F \in H^2(M, \mathbb{Z})$ while H gives a class in $H^3(E, \mathbb{Z})$

The T-dual of (E, H) is given by the pair $(\widehat{E}, \widehat{H})$, where the principal S^1-bundle

$$\begin{array}{ccc} \widehat{S}^1 & \longrightarrow & \widehat{E} \\ & & \Big\downarrow \widehat{\pi} \\ & & M \end{array}$$

and the dual H-flux $\widehat{H} \in H^3(\widehat{E}, \mathbb{Z})$, satisfy

$$\boxed{\widehat{F} = \pi_* H \,, \qquad F = \widehat{\pi}_* \widehat{H}}$$

where $\pi_* : H^3(E, \mathbb{Z}) \to H^2(M, \mathbb{Z})$, and $\widehat{\pi}_* : H^3(\widehat{E}, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ are the pushforward maps ('integration over the S^1-fiber')

T-duality for principal S^1-bundles (cont'd)

$$
\begin{array}{ccccc}
 & & E \times_M \widehat{E} & & \\
 & \swarrow^{p=1\times\widehat{\pi}} & & \searrow^{\widehat{p}=\pi\times 1} & \\
E & & & & \widehat{E} \\
 & \searrow_{\pi} & & \swarrow_{\widehat{\pi}} & \\
 & & M & &
\end{array}
$$

T-duality for principal S^1-bundles (cont'd)

The ambiguity in the choice of $\widehat{H}$ is removed by requiring that

$$p^* H - \widehat{p}^* \widehat{H} \equiv 0$$

in $H^3(E \times_M \widehat{E}, \mathbb{Z})$, where $E \times_M \widehat{E}$ is the correspondence space

$$E \times_M \widehat{E} = \{(x, \widehat{x}) \in E \times \widehat{E} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$$

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Theorem (B-Evslin-Mathai)

This T-duality gives rise to an isomorphism between the twisted cohomologies and twisted K-theories of (E, H) and $(\widehat{E}, \widehat{H})$ (with a shift in degree by 1)

T-duality and generalized geometry

Given a principal circle bundle E with H-flux $H \in \Omega^3_{\rm cl}(E)^{S^1}$

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \big\downarrow \pi \\ & & M \end{array} \qquad H = H_{(3)} + A \wedge H_{(2)}\,, \;\; F = dA$$

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there exists a T-dual principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \widehat{E} \\ & & \downarrow \widehat{\pi} \\ & & M \end{array} \qquad \widehat{H} = H_{(3)} + \widehat{A} \wedge F, \; \widehat{F} = H_{(2)} = d\widehat{A}$$

Theorem [B-Evslin-Mathai, Cavalcanti-Gualtieri]

(a) We have an isomorphism of differential complexes $\tau : (\Omega^\bullet(E)^{S^1}, d_H) \to (\Omega^\bullet(\widehat{E})^{S^1}, d_{\widehat{H}})$

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$$\tau(\Omega_{(k)} + A \wedge \Omega_{(k-1)}) = -\Omega_{(k-1)} + \widehat{A} \wedge \Omega_{(k)}$$
$$\tau \circ d_H = -d_{\widehat{H}} \circ \tau$$

T-duality and generalized geometry (cont'd)

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Hence, τ induces an isomorphism on twisted cohomology

(b) We can identify $(X + \Xi \in \Gamma(TE \oplus T^*E)^{S^1}$ with a quadruple $(x, f; \xi, g)$

$$X = x + f\partial_A\,, \qquad \Xi = \xi + gA$$

and define a map $\phi : \Gamma(TE \oplus T^*E)^{S^1} \to \Gamma(T\widehat{E} \oplus T^*\widehat{E})^{S^1}$

$$\phi(x + f\partial_A + \xi + gA) = x + g\partial_{\widehat{A}} + \xi + f\widehat{A}$$

T-duality and generalized geometry (cont'd)

(b) The map ϕ is orthogonal wrt pairing on $TE \oplus T^*E$, hence
τ induces an isomorphism of Clifford algebras

T-duality and generalized geometry (cont'd)

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(c) For $X + \Xi \in \Gamma((TE \oplus T^*E)^{S^1})$ we have

$$\tau(\gamma_{X+\Xi} \cdot \Omega) = \gamma_{\phi(X+\Xi)} \cdot \tau(\Omega)$$

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(d) For $X_i + \Xi_i \in \Gamma((TE \oplus T^*E)^{S^1})$ we have

$$\phi\left([\![X_1 + \Xi_1, X_2 + \Xi_2]\!]_H\right) = [\![\phi(X_1 + \Xi_1), \phi(X_2 + \Xi_2)]\!]_{\widehat{H}}$$

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Hence ϕ gives a homomorphism of twisted Courant brackets

It follows that T-duality acts naturally on generalized complex structures, generalized Kähler structures, generalized Calabi-Yau structures, ...

Dimensionally reduced Dorfman bracket

The dimensionally reduced Dorfman bracket

$$
\begin{aligned}
&(x_1, f_1; \xi_1, g_1) \circ (x_2, f_2; \xi_2, g_2) = \\
&\quad ([x_1, x_2], x_1(f_2) - x_2(f_1) + \imath_{x_1}\imath_{x_2} F; \\
&\quad (\mathcal{L}_{x_1}\xi_2 - \imath_{x_2} d\xi_1) + \imath_{x_1}\imath_{x_2} H_{(3)} + (df_1 g_2 + f_2 dg_1) \\
&\quad + (g_2 \imath_{x_1} F - g_1 \imath_{x_2} F) + (f_2 \imath_{x_1} H_{(2)} - f_1 \imath_{x_2} H_{(2)}), \\
&\quad x_1(g_2) - x_2(g_1) + \imath_{x_1}\imath_{x_2} H_{(2)})
\end{aligned}
$$

is that of the transitive Courant algebroid $E = TM \oplus (\mathfrak{t} \oplus \mathfrak{t}^*) \oplus T^*M$ with $R = -(F, H_{(2)})$, $H = -H_{(3)}$ and $\langle\ ,\ \rangle_{\mathfrak{G}}$ the canonical pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$.

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&+ (g_2\imath_{x_1}F - g_1\imath_{x_2}F) + (f_2\imath_{x_1}H_{(2)} - f_1\imath_{x_2}H_{(2)}), \\
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[Doubling of the Atiyah algebroid corresponding to the principal S^1-bundle]

Generalization to principal torus bundles

We have

$$H = H_{(3)} + A_i \wedge H^i_{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)}$$

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such that

$$\begin{aligned} d_H = & \bar{d} + H_{(3)} + F_{(2)i} \partial_{A_i} \\ & + A_i \wedge H^i_{(2)} + \tfrac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \tfrac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)} \end{aligned}$$

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The $F_{(1)ij}$ and $F_{(0)ijk}$ are known as nongeometric fluxes

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Theorem (B-Garretson-Kao)

T-duality provides an isomorphism of (certain) Courant algebroids

THANKS