Topological defects and Khovanov-Rozansky homology

based on arXiv:1006.5609 [hep-th] with **Ingo Runkel**, and work with **Daniel Murfet**

Nils Carqueville

LMU München

Topological defects

A 2-dimensional **topological field theory with defects** is a symmetric monoidal functor

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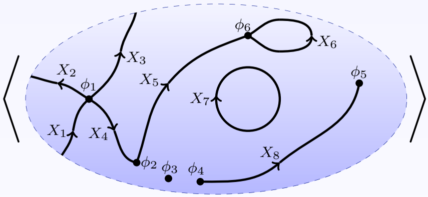

independent of position of defect lines

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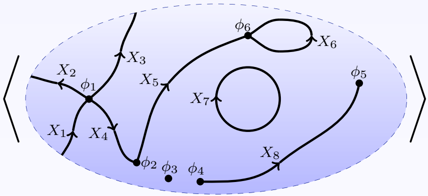

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Affine **Landau-Ginzburg models**: defects and defect fields between W_1 and W_2 described by $\mathrm{MF}(W_1 \otimes 1 - 1 \otimes W_2)$

Kapustin/Rozansky 2004, Brunner/Roggenkamp 2007

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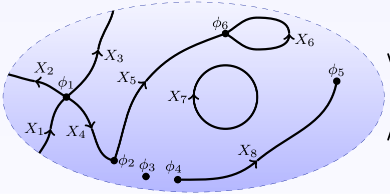
$\Big\langle$ 

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Example. Invisible defect

$$I = \begin{pmatrix} 0 & x-y \\ \frac{W(x)-W(y)}{x-y} & 0 \end{pmatrix} \in \mathrm{MF}(W \otimes 1 - 1 \otimes W)$$

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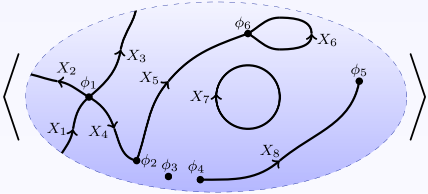

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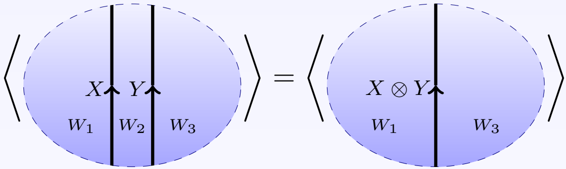

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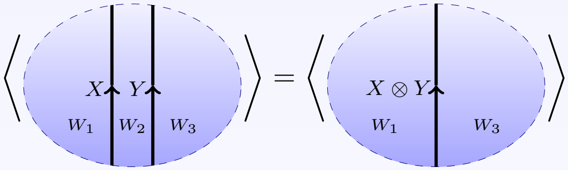

Theorem. $\mathrm{MF}(W \otimes 1 - 1 \otimes W)$ is monoidal.

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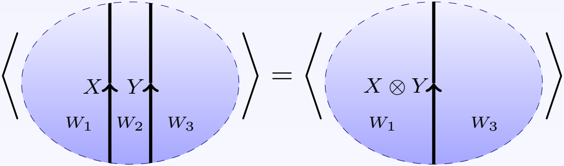

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$(X \otimes Y)$

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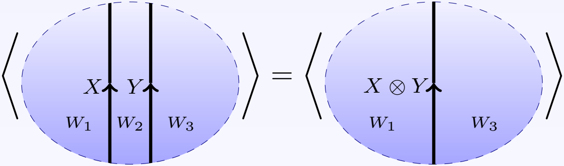

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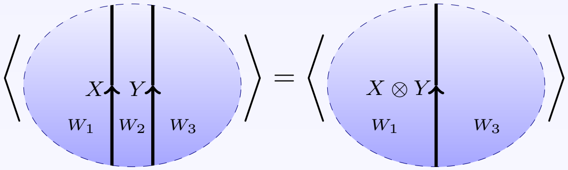

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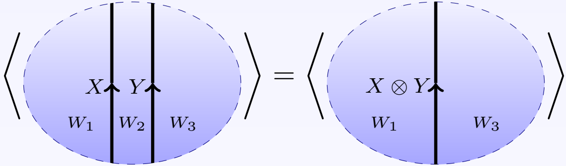

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Remark. Can compute $6j$-symbols, agreement with rational CFT.

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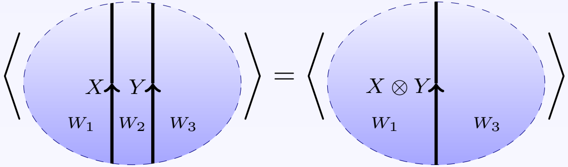

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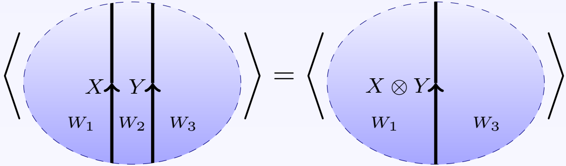

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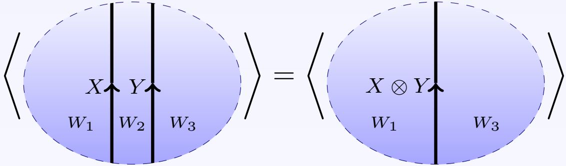

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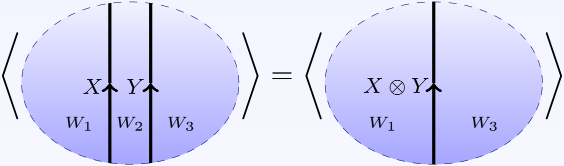

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- 2-morphisms: morphisms in $\mathrm{MF}(W_1 \otimes 1 - 1 \otimes W_2)$

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Action on bulk fields

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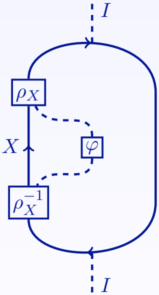
$\longmapsto \mathcal{D}_r(X)(\varphi) =$ 

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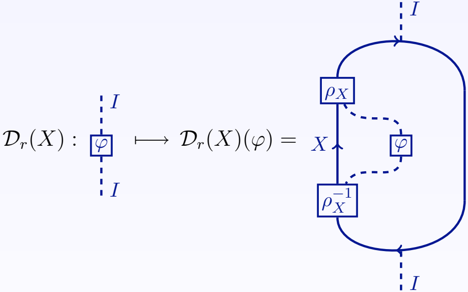
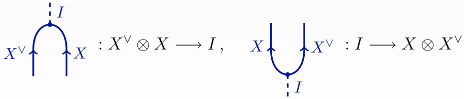


Need evaluation and coevaluation maps:



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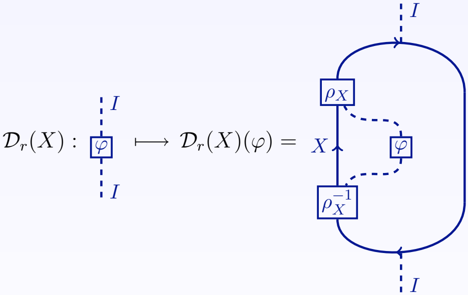

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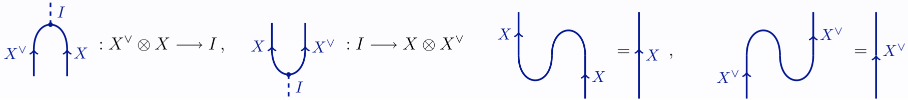

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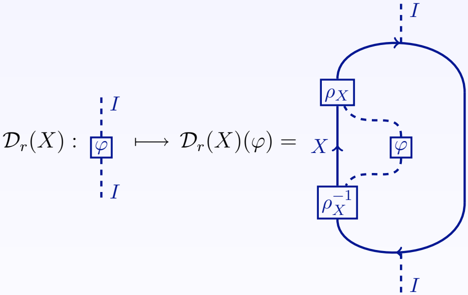

Theorem. $\mathrm{MF}(W \otimes 1 - 1 \otimes W)$ is pivotal rigid monoidal

Carqueville/Runkel 2010

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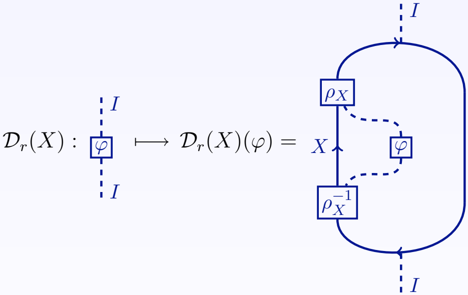

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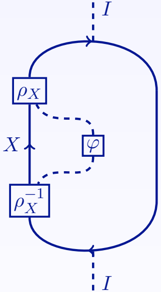

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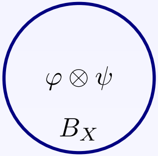
Theorem. $\mathcal{D}_l$, $\mathcal{D}_r$ induce ring (anti-)homomorphisms

$$K_0(\mathrm{MF}(W \otimes 1 - 1 \otimes W)) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{End}^0(\mathrm{End}_{\mathrm{MF}(W\otimes 1 - 1 \otimes W)}(I))$$

Carqueville/Runkel 2010

Action on bulk fields

$$\left\langle \mathcal{D}_l(X)(\varphi)\,\psi \right\rangle_{\mathsf{bulk}} = \left\langle \quad \right\rangle$$



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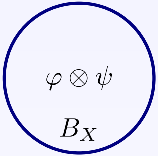

$$\Bigg\rangle$$

$$\mathcal{D}_l(X)(\varphi) = \mathrm{Res}\left[\frac{\varphi\ \mathrm{str}\,(\partial_{x_1}X\partial_{y_1}X\ldots\partial_{y_N}X)\,\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_N}{\partial_{x_1}W\ldots\partial_{x_N}W}\right]$$

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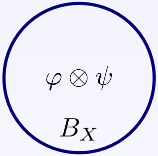

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Proposition. For all known defects of $W = x^d$ we have

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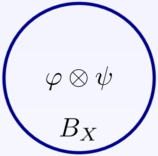

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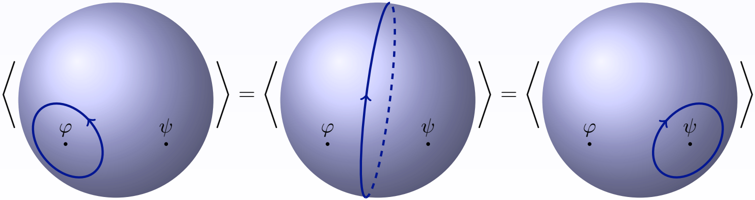

Khovanov-Rozansky link invariants

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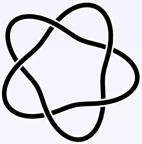
In any **link diagram** 

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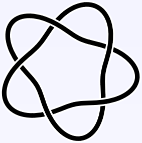
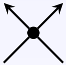
In any **link diagram**  replace crossing by  or 

Khovanov/Rozansky 2004

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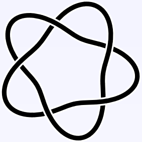
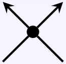
In any **link diagram**  replace crossing by  or 

$\Rightarrow$ get special planar graph.

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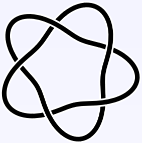
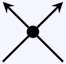
In any **link diagram**  replace crossing by  or 

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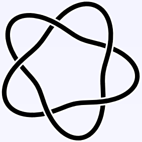
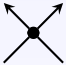
In any **link diagram**  replace crossing by  or 

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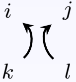
variable x_n to edge n, $X_\circ$ to 

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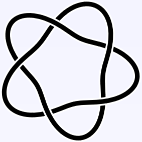
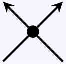
In any **link diagram**  replace crossing by  or 

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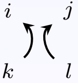
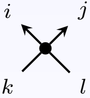
variable x_n to edge n, $X_\circ$ to 

, $X_\bullet$ to 

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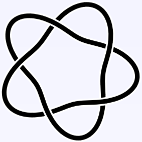
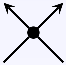
In any **link diagram**  replace crossing by  or 

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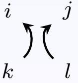
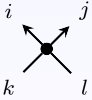
variable x_n to edge n, $X_\circ$ to 

, $X_\bullet$ to 

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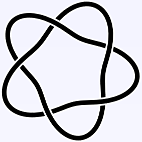
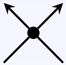
Khovanov/Rozansky 2004

Khovanov-Rozansky link invariants

In any **link diagram**  replace crossing by  or 

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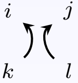
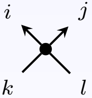
variable x_n to edge n, $X_\circ$ to 

, $X_\bullet$ to 

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Khovanov/Rozansky 2004

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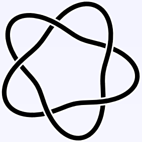
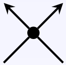
In any **link diagram**  replace crossing by  or 

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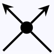
, $X_\bullet$ to 

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Khovanov/Rozansky 2004

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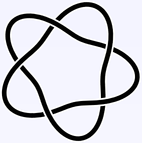
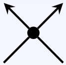
In any **link diagram**  replace crossing by  or 

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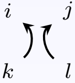
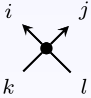
variable x_n to edge n , $X_\circ$ to 

, $X_\bullet$ to 

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Khovanov/Rozansky 2004

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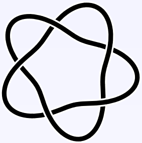
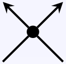
In any **link diagram**  replace crossing by  or 

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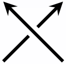
Associate complexes of matrix factorisations to crossings in link diagram:

 $\widehat{=} \left(\underline{X_\circ\{1-n\}} \xrightarrow{\chi_0} X_\bullet\{-n\} \right)$

Khovanov/Rozansky 2004

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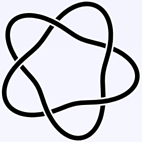
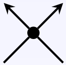
In any **link diagram**  replace crossing by  or 

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Khovanov/Rozansky 2004

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Khovanov/Rozansky 2004

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Khovanov/Rozansky 2004

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Khovanov/Rozansky 2004

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Theorem. There is an idempotent on $X \otimes Y$ whose splitting is T.

Khovanov/Rozansky 2004, Dyckerhoff/Murfet 2011

Computing Khovanov-Rozansky link invariants

Implementation of idempotent splitting in *Singular*

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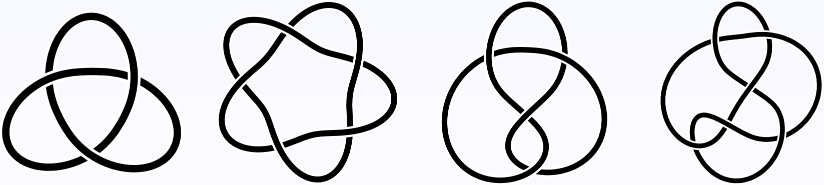

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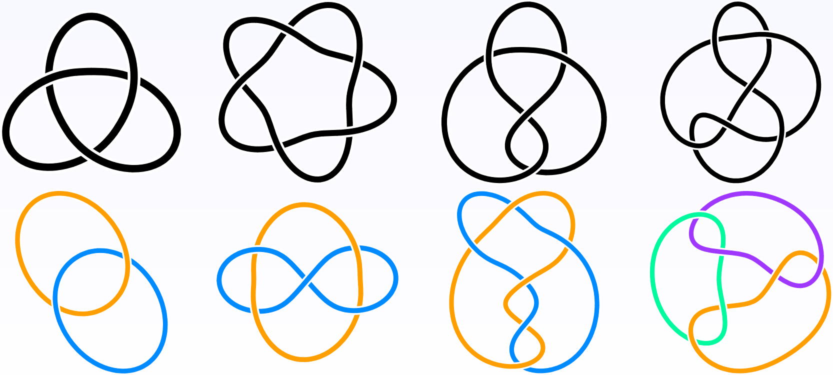

Carqueville/Murfet 2011

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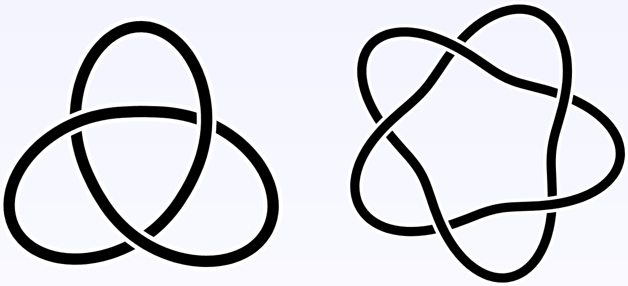
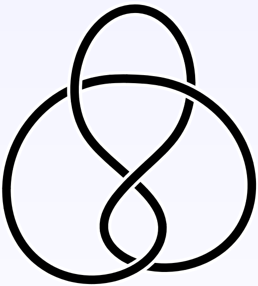


$$\begin{aligned}\mathrm{KR}_8 = (&q^{39} + q^{37} + q^{35} + q^{33} + q^{31} + q^{29} + q^{27} + q^{23}t + q^{21}t^3 + q^{21}t\\ &+ q^{19}t^3 + q^{19}t + q^{17}t^3 + q^{17}t + q^{15}t^3 + q^{15}t + q^{13}t^3 + q^{13}t\\ &+ q^{11}t^3 + q^{11}t + q^9t^3 + q^7t^3)/t^3\end{aligned}$$

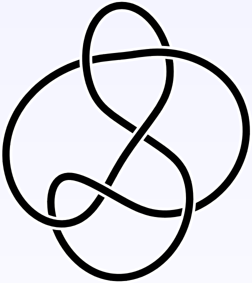
$$\begin{aligned}\mathrm{KR}_6 = (&q^{28} + q^{26} + q^{24}t^2 + q^{24} + q^{22}t^2 + q^{22} + q^{20}t^4 + q^{20}t^2 + q^{20}\\ &+ q^{18}t^4 + q^{18}t^2 + q^{18} + q^{16}t^4 + q^{16}t^2 + q^{14}t^4 + q^{12}t^4 + q^{12}t^3\\ &+ q^{10}t^3 + q^8t^5 + q^8t^3 + q^6t^5 + q^6t^3 + q^4t^5 + q^4t^3 + q^2t^5 + t^5)/q^{43}\end{aligned}$$

Computing Khovanov-Rozansky link invariants



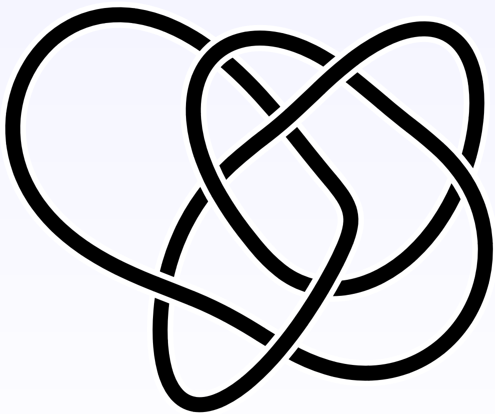
$$\mathrm{KR}_5 = (q^{28} + q^{26} + q^{24} + q^{22} + q^{18}t^2 + q^{18}t + q^{16}t^3 + q^{16}t^2 + q^{16}t + q^{14}t^3 + q^{14}t^2 + q^{14}t + q^{12}t^3 + q^{12}t^2 + q^{12}t + q^{10}t^3 + q^{10}t^2 + q^6t^4 + q^4t^4 + q^2t^4 + t^4)/(q^{14}t^2)$$

Computing Khovanov-Rozansky link invariants



$$\begin{aligned}\mathrm{KR}_3 = (&q^{20} + q^{18}t + q^{18} + q^{16}t^2 + q^{16}t + q^{14}t^2 + q^{14}t + 2q^{12}t^3 \\ &+ q^{12}t^2 + q^{12}t + q^{10}t^4 + 3q^{10}t^3 + q^{10}t^2 + q^8t^5 + q^8t^4 \\ &+ 2q^8t^3 + q^6t^5 + q^6t^4 + q^4t^5 + q^4t^4 + q^2t^6 + q^2t^5 + t^6)/(q^{10}t^3)\end{aligned}$$

Computing Khovanov-Rozansky link invariants



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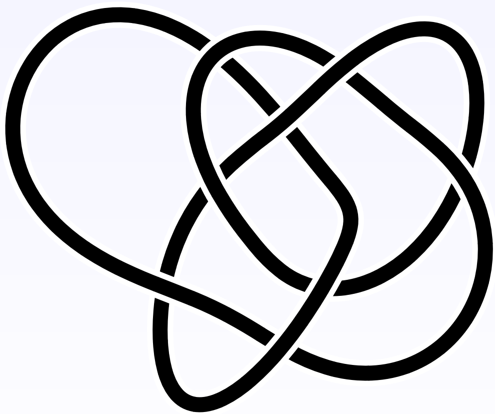


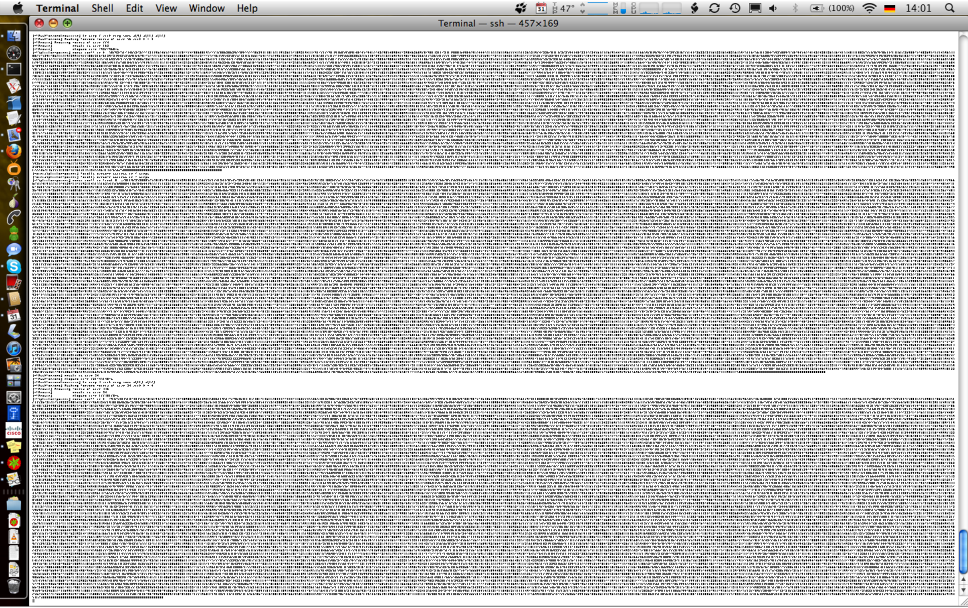
$$\mathrm{KR}_2 = (t^5q^7 + t^5q^5 + t^3q^9 + t^2q^{13} + tq^{13} + tq^{11} + q^{17} + q^{15})/(t^5)$$

Terminal Shell Edit View Window Help

Terminal — bash — 228×60

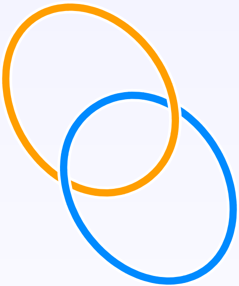
```
[webCompile] even grading: 6,4,2,0,4,2,0,-2,2,0,-2,-4,0,-2,-4,-6
[webCompile]  odd grading: 0
[linkCompile] Saving to disk.

[linkCompile] Looking at state [1,0] which is number [[ 2/4 ]].
[linkCompile] Currently used memory: 850kb.
[webCompile] Entering round 1 of compilation.
[mfPushforwardInductive] In step 1 with ring vars x(1),x(2),x(3),x(4)
[mfPushforward] Pushing forward matrix of size 16 with N = 4
[mfReduce] Reducing matrix of size 64
[mfReduce]          result is size 16.
[mfReduce]          elapsed time 830ms.
[mfSplitIdempotent] Badness of worst coeff in E: 5
[matrixSplitIdempotent] Split idempotent matrix in 3 steps.
[matrixSplitIdempotent] Split idempotent matrix in 3 steps.
[mfSplitIdempotent] Badness of worst coeff in final B: 3
[mfSplitIdempotent] Worst coeff in final B: -10
[mfPushforward] result is size 8.
[mfPushforward] elapsed time 2800ms.
[mfPushforwardInductive] In step 2 with ring vars x(2),x(3),x(4)
[mfPushforward] Pushing forward matrix of size 8 with N = 4
[mfReduce] Reducing matrix of size 32
[mfReduce]          result is size 8.
[mfReduce]          elapsed time 200ms.
[mfSplitIdempotent] Badness of worst coeff in E: 5
[matrixSplitIdempotent] Split idempotent matrix in 3 steps.
[matrixSplitIdempotent] Split idempotent matrix in 3 steps.
[mfSplitIdempotent] Badness of worst coeff in final B: 3
[mfSplitIdempotent] Worst coeff in final B: 1/5
[mfPushforward] result is size 4.
[mfPushforward] elapsed time 660ms.
[mfPushforwardInductive] In step 3 with ring vars x(3),x(4)
[mfPushforward] Pushing forward matrix of size 4 with N = 4
[mfReduce] Reducing matrix of size 16
[mfReduce]          result is size 12.
[mfReduce]          elapsed time 50ms.
[mfSplitIdempotent] Badness of worst coeff in E: 4
[matrixSplitIdempotent] Split idempotent matrix in 5 steps.
[matrixSplitIdempotent] Split idempotent matrix in 2 steps.
[mfSplitIdempotent] Badness of worst coeff in final B: 1
[mfSplitIdempotent] Worst coeff in final B: 5
[mfPushforward] result is size 6.
[mfPushforward] elapsed time 200ms.
[mfPushforwardInductive] In step 4 with ring vars x(4)
[mfPushforward] Pushing forward matrix of size 6 with N = 4
[mfReduce] Reducing matrix of size 24
[mfReduce]          result is size 24.
[mfReduce]          elapsed time 60ms.
[mfSplitIdempotent] Badness of worst coeff in E: 1
[matrixSplitIdempotent] Split idempotent matrix in 0 steps.
[mfPushforward] elapsed time 240ms.
[mfPushforwardInductive] total elapsed time 3910ms.
[webCompile] total elapsed time 4060ms.
[webCompile] even grading: 1,-1,-3,-5,5,3,1,-1,3,1,-1,-3
[webCompile]  odd grading: 0
[linkCompile] Saving to disk.

[linkCompile] Looking at state [0,1] which is number [[ 3/4 ]].
[linkCompile] Currently used memory: 1415kb.
```


Terminal Shell Edit View Window Help
09:47
Terminal — ssh — 228×60
[mfPushforward] result is size 84.
[mfPushforward] elapsed time 281745140ms.
[mfPushforwardInductive] In step 3 with ring vars x(11),x(12)
[mfPushforward] Pushing forward matrix of size 84 with N = 4
[mfReduce] Reducing matrix of size 336


Terminal
Shell
Edit
View
Window
Help
47°
(100%)
14:01
Terminal — ssh — 457×169


Terminal Shell Edit View Window Help
44°
(96%)
08:23
Terminal — ssh — 457×169

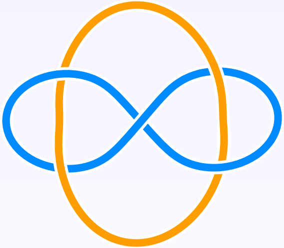
Computing Khovanov-Rozansky link invariants



$$\mathrm{KR}_2 = (q^6 + q^4 + q^2t^2 + t^2)/t^2$$

$$\begin{aligned}\mathrm{KR}_{11} = (&q^{42} + 2q^{40} + 3q^{38} + 4q^{36} + 5q^{34} + 6q^{32} + 7q^{30} + 8q^{28} + 9q^{26} \\ &+ 10q^{24} + 10q^{22} + q^{20}t^2 + 9q^{20} + q^{18}t^2 + 8q^{18} + q^{16}t^2 \\ &+ 7q^{16} + q^{14}t^2 + 6q^{14} + q^{12}t^2 + 5q^{12} + q^{10}t^2 + 4q^{10} + q^8t^2 \\ &+ 3q^8 + q^6t^2 + 2q^6 + q^4t^2 + q^4 + q^2t^2 + t^2)/t^2\end{aligned}$$

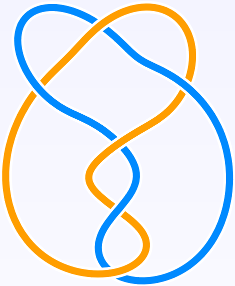
Computing Khovanov-Rozansky link invariants



$$\begin{aligned}\mathrm{KR}_5 = (&q^{24} + q^{22} + q^{20}t^2 + q^{20} + 2q^{18}t^2 + 4q^{16}t^2 + q^{16}t + q^{14}t^3 \\ &+ 4q^{14}t^2 + q^{14}t + q^{12}t^4 + q^{12}t^3 + 3q^{12}t^2 + q^{12}t \\ &+ q^{10}t^4 + q^{10}t^3 + 2q^{10}t^2 + q^8t^4 + q^6t^4 + q^4t^5 + q^4t^4 \\ &+ q^2t^5 + q^2t^4 + t^5)/(q^{16}t^2)\end{aligned}$$

Carqueville/Murfet 2011

Computing Khovanov-Rozansky link invariants



$$\mathrm{KR}_4 = (q^{20} + q^{18}t + q^{16}t^2 + q^{14}t^3 + q^{12}t^2 + q^{10}t^3 + 2q^8t^4 + q^4t^6 + q^2t^6 + q^2t^5 + t^6)/q^{32}$$

Computing Khovanov-Rozansky link invariants

$$\mathrm{KR}_3 = (q^{16} + 2q^{14}t + q^{12}t + 2q^{10}t^3 + 2q^{10}t^2 + 5q^8t^3 + 2q^6t^4 + 2q^6t^3 + q^4t^5 + 2q^2t^5 + t^6)/q^8t^3$$

Computing Khovanov-Rozansky link invariants

$$\mathrm{KR}_3 = (q^{16} + 2q^{14}t + q^{12}t + 2q^{10}t^3 + 2q^{10}t^2 + 5q^8t^3 + 2q^6t^4 + 2q^6t^3 + q^4t^5 + 2q^2t^5 + t^6)/q^8t^3)$$

$$\mathrm{KR}_3' = (q^{12} + q^{10} + q^8t^2 + 3q^6t^2 + q^4t^2 + q^2t^4 + t^4)/(q^6t^2)$$

Computing Khovanov-Rozansky link invariants

$$\mathrm{KR}_3 = (q^{16} + 2q^{14}t + q^{12}t + 2q^{10}t^3 + 2q^{10}t^2 + 5q^8t^3 + 2q^6t^4 + 2q^6t^3 + q^4t^5 + 2q^2t^5 + t^6)/q^8t^3)$$

$$\mathrm{KR}_3' = (q^{12} + q^{10} + q^8t^2 + 3q^6t^2 + q^4t^2 + q^2t^4 + t^4)/(q^6t^2)$$

$$\mathrm{KR}_4 = (q^{20} + 2q^{18}t + 2q^{14}t^3 + q^{14}t + 3q^{12}t^3 + 2q^{12}t^2 + 6q^{10}t^3 + 2q^8t^4 + 3q^8t^3 + q^6t^5 + 2q^6t^3 + 2q^2t^5 + t^6)/(q^{10}t^3)$$