# Topological defects and Khovanov-Rozansky homology

based on arXiv:1006.5609 [hep-th] with Ingo Runkel, and work with Daniel Murfet

**Nils Carqueville** 

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Affine Landau-Ginzburg models: defects and defect fields between  $W_1$ and  $W_2$  described by  $MF(W_1 \otimes 1 - 1 \otimes W_2)$ 

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$$I = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix} \in \mathrm{MF}(W \otimes 1 - 1 \otimes W)$$

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$$\left\langle \left( \begin{array}{c} X \\ W_1 \end{array} \right) \\ W_2 \end{array} \right\rangle = \left\langle \left( \begin{array}{c} X \otimes Y \\ W_1 \end{array} \right) \\ W_3 \end{array} \right\rangle$$

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**Theorem.**  $\mathcal{D}_l$ ,  $\mathcal{D}_r$  induce ring (anti-)homomorphisms  $K_0(\mathrm{MF}(W \otimes 1 - 1 \otimes W)) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{End}^0(\mathrm{End}_{\mathrm{MF}(W \otimes 1 - 1 \otimes W)}(I))$ 

$$\left\langle \mathcal{D}_l(X)(\varphi) \psi \right\rangle_{\mathsf{bulk}} = \left\langle \left( \begin{array}{c} \varphi \otimes \psi \\ B_X \end{array} \right) \right\rangle$$

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**Proposition.** For all known defects of  $W = x^d$  we have  $\langle \mathcal{D}_l(X)(\varphi) \psi \rangle = \langle \varphi \mathcal{D}_r(X)(\psi) \rangle$ 

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variable  $x_n$  to edge n














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**Theorem.**  $H(\mathcal{L})$  is a link invariant and concentrated in one  $\mathbb{Z}_2$ -degree.

$$\operatorname{KR}_{n}(\mathcal{L}) = \sum_{i,j \in \mathbb{Z}} t^{i} q^{j} \operatorname{dim}_{\mathbb{Q}}(H^{i,j}(\mathcal{L}))$$

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Implementation of idempotent splitting in Singular

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$$\begin{split} \mathrm{KR}_8 &= (q^{39} + q^{37} + q^{35} + q^{33} + q^{31} + q^{29} + q^{27} + q^{23}t + q^{21}t^3 + q^{21}t \\ &+ q^{19}t^3 + q^{19}t + q^{17}t^3 + q^{17}t + q^{15}t^3 + q^{15}t + q^{13}t^3 + q^{13}t \\ &+ q^{11}t^3 + q^{11}t + q^9t^3 + q^7t^3)/t^3 \\ \mathrm{KR}_6 &= (q^{28} + q^{26} + q^{24}t^2 + q^{24} + q^{22}t^2 + q^{22} + q^{20}t^4 + q^{20}t^2 + q^{20} \\ &+ q^{18}t^4 + q^{18}t^2 + q^{18} + q^{16}t^4 + q^{16}t^2 + q^{14}t^4 + q^{12}t^4 + q^{12}t^3 \\ &+ q^{10}t^3 + q^8t^5 + q^8t^3 + q^6t^5 + q^6t^3 + q^4t^5 + q^4t^3 + q^2t^5 + t^5)/q^{43} \end{split}$$



 $\begin{aligned} \mathrm{KR}_5 &= (q^{28} + q^{26} + q^{24} + q^{22} + q^{18}t^2 + q^{18}t + q^{16}t^3 + q^{16}t^2 + q^{16}t \\ &+ q^{14}t^3 + q^{14}t^2 + q^{14}t + q^{12}t^3 + q^{12}t^2 + q^{12}t + q^{10}t^3 \\ &+ q^{10}t^2 + q^6t^4 + q^4t^4 + q^2t^4 + t^4)/(q^{14}t^2) \end{aligned}$ 



$$\begin{split} \mathrm{KR}_3 &= (q^{20} + q^{18}t + q^{18} + q^{16}t^2 + q^{16}t + q^{14}t^2 + q^{14}t + 2q^{12}t^3 \\ &+ q^{12}t^2 + q^{12}t + q^{10}t^4 + 3q^{10}t^3 + q^{10}t^2 + q^8t^5 + q^8t^4 \\ &+ 2q^8t^3 + q^6t^5 + q^6t^4 + q^4t^5 + q^4t^4 + q^2t^6 + q^2t^5 + t^6)/(q^{10}t^3) \end{split}$$





$$\mathrm{KR}_2 = (t^5 q^7 + t^5 q^5 + t^3 q^9 + t^2 q^{13} + tq^{13} + tq^{11} + q^{17} + q^{15})/(t^5)$$

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-	[vebCompile] Entering round 1 of compilation.		
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1	Informational polytics and the formation of the formation		
119	[mfReduce] result is size 16.		
	[mfReduce] elopsed time 830ms.		
3	[mfSplitIdempotent] Badness of worst coeff in E: 5		
Ă	[motrisplitidempotent] split idempotent motrix in 3 steps.		
X	Infortidempotent Badness of worst coeff in final B: 3		
714	[mfSplitIdempotent] Worst coeff in final B: -10		
6	[mfPushforward] result is size 8.		
a	[mfPushforward] elopsed time 2000ms. [mfPushforward] to star 2 with sing ware w(2) w(2) w(4)		
0	[in Postin Ground Lindicated] in step 2 with ring was $x(2)/x(3)/x(3)$ Definition and Dubling forward active of size 8 with N = 4		
	[mfReduce] Reducing matrix of size 32		
	[mfReduce] result is size 8.		
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5	[wisplitidempotent] Boards or Worst coerr in E: 5 [wisplitidempotent] Bill idemotent activity in 3 stere		
	[matrixSplitIdempotent] Split idempotent matrix in 3 steps.		
	[mfSplitIdempotent] Badness of vorst coeff in final B: 3		
	[mfSplitIdempotent] Worst coeff in final B: 1/5		
15	Whyshororward result is size +.		
1	[mPushforwardInductive] In step 3 with ring vars x(3),x(4)		'n
2	[mfPushforward] Pushing forward matrix of size 4 with N = 4		
U	[mfReduce] Reducing matrix of size 16		
70	[Witeduce] Pesuit is size 12.		
	[mfSplitIdempotent] Badness of worst coeff in E: 4		
	[watrixSplitIdempotent] Split idempotent matrix in 5 steps.		
0	[matrixSplitIdempotent] Split idempotent matrix in 2 steps.		
9	[wisplitidempotent] Badness of Worst coeff in final B: 1		
	Influcture result is size 6.		
cisco	[mfPushforward] elapsed time 200ms.		
1	[mfPushforwardInductive] In step 4 with ring vars x(4)		
	[MPAshforward] Pushing forward matrix of size 5 with N = 4 [MPashforward] Pushing water of size 24		
2	Infreduce results size 24.		
88	[infReduce] elapsed time 60ms.		
ЩШ	[wfSplitIdempotent] Badness of vorst coeff in E: 1		
	[metrixSpitiempotent] Spiti tempotent mutrix in e stepS.		
1	Infrustrovanding contactive total elassed time 3918ms.		
	[vebCompile] total elapsed time 4060ms.		
ė.	[vebCompile] even grading: 1,-1,-3,-5,5,3,1,-1,3,1,-1,-3		
0	[NebCompile] dad grading: d		
	Frincombined second on erect		J
	[linkCompile] Looking at state [0,1] which is number [[ 3/4 ]].		ž
101	[linkCompile] Currently used memory: 1415kb.		ŕ

Terminal - ssh - 228×60 9457221215561338323318164210164520x4373938719185883544366697849141182642925165722449635119558548117699217220493381888866545561452933884328988741792784572139584887146688944833165736493151297875654075990798376449649819362342624 N746252442214622696555484336272056637144787932142860169324528973876984117667599703881065852971217930844637051397522277617553176938321476918837684240970938668528126471893481066670508321238773867966749124816831939685845452297321 67064459X77217234488114455580739222983338663195507578565978565978994493152278659999493152278657298822786199255678564396557514396557578643965572155249398229183238567819457229882291841832229856578923144522298556578982278619827774990 572%1 0066651 768878201 45822060 366624 3847631 8437281 7085872 385 1656274 21 8537372000082518 328544 3564 554245 14853727201 460046531 727081 986528028 3275384 3389687564 20285208455244 254455444 5545524 157584 15757201 450046531 727081 986528028 3275384 3389687564 20285208453544 254455444 5545524 157584 1575724 157584 1575724 1575724 1575724 1575724 1575724 1575724 1575724 1575724 1575724 1575774 1575774 1575774 1575774 15757774 15757774 15757 31707/65/45947726721548329265/749907581514556751749453269377476557182588366559396733211082725166629129127815817187837288157701610%36183835888228895954016774714388585215859617972196327927315820880311816265755388866255818208822989 1873797326124211795894464518492665232729374391119559110377431108224988765987126559476032883998581643137854840477381455811436641474758455363153 3626272993685558817302566760333814667177246202700856187725128568164077615814 8557/461667/k1123238962878697/k998857/k9988727965625091372015218311631949158281164733898273562387897797/895805517188286765588449/61692663725288667811828277048185356114149481328889354676641898955683454189895868377 333311 276666 38792149 1984982941 97569146 453316293736745 3407615451625279884 25848537383227592145898185902 34732227838745868454 76165237186154734418657178156 885677881 88844 3817258933383128891 9872865755371845533189571 199312045773 24525661488408579776648794912218x617336418794262348159828571914018988488277749463014959028878415563854304939063635221486781914628515473159383122335674810779796313838801537272498894632611675273934233218712942958822777418965450 16611563355338128634241628553157513847783965348396450264964781338682724933387855779168341739777631705006626344286578000051443311549724931597784783380491859735328885120000312434808757514700726165131094542893826546/0516442 [mfPushforward] result is size 84. [nfPushforward] elapsed time 281745148ms. [mfPushforwardInductive] In step 3 with ring vars x(11),x(12) afPushforward] Pushing forward matrix of size 84 with N = 4 infReduce1 Reducing matrix of size 336

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#### 📹 Terminal Shell Edit View Window Help

#### Terminal - ssh - 457×169

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$$\begin{aligned} \mathrm{KR}_2 &= (q^6 + q^4 + q^2 t^2 + t^2)/t^2 \\ \mathrm{KR}_{11} &= (q^{42} + 2q^{40} + 3q^{38} + 4q^{36} + 5q^{34} + 6q^{32} + 7q^{30} + 8q^{28} + 9q^{26} \\ &\quad + 10q^{24} + 10q^{22} + q^{20}t^2 + 9q^{20} + q^{18}t^2 + 8q^{18} + q^{16}t^2 \\ &\quad + 7q^{16} + q^{14}t^2 + 6q^{14} + q^{12}t^2 + 5q^{12} + q^{10}t^2 + 4q^{10} + q^8t^2 \\ &\quad + 3q^8 + q^6t^2 + 2q^6 + q^4t^2 + q^4 + q^2t^2 + t^2)/t^2 \end{aligned}$$



$$\begin{aligned} \mathrm{KR}_5 &= (q^{24} + q^{22} + q^{20}t^2 + q^{20} + 2q^{18}t^2 + 4q^{16}t^2 + q^{16}t + q^{14}t^3 \\ &+ 4q^{14}t^2 + q^{14}t + q^{12}t^4 + q^{12}t^3 + 3q^{12}t^2 + q^{12}t \\ &+ q^{10}t^4 + q^{10}t^3 + 2q^{10}t^2 + q^8t^4 + q^6t^4 + q^4t^5 + q^4t^4 \\ &+ q^2t^5 + q^2t^4 + t^5)/(q^{16}t^2) \end{aligned}$$



# $$\begin{split} \mathrm{KR}_4 &= (q^{20} + q^{18}t + q^{16}t^2 + q^{14}t^3 + q^{12}t^2 + q^{10}t^3 + 2q^8t^4 + q^4t^6 \\ &\quad + q^2t^6 + q^2t^5 + t^6)/q^{32} \end{split}$$



# $$\begin{split} \mathrm{KR}_3 &= (q^{16} + 2q^{14}t + q^{12}t + 2q^{10}t^3 + 2q^{10}t^2 + 5q^8t^3 + 2q^6t^4 + 2q^6t^3 \\ &\quad + q^4t^5 + 2q^2t^5 + t^6)/q^8t^3 \end{split}$$



$$\begin{split} \mathrm{KR}_3 &= (q^{16} + 2q^{14}t + q^{12}t + 2q^{10}t^3 + 2q^{10}t^2 + 5q^8t^3 + 2q^6t^4 + 2q^6t^3 \\ &\quad + q^4t^5 + 2q^2t^5 + t^6)/q^8t^3) \\ \mathrm{KR}_3' &= (q^{12} + q^{10} + q^8t^2 + 3q^6t^2 + q^4t^2 + q^2t^4 + t^4)/(q^6t^2) \end{split}$$



$$\begin{split} \mathrm{KR}_3 &= (q^{16} + 2q^{14}t + q^{12}t + 2q^{10}t^3 + 2q^{10}t^2 + 5q^8t^3 + 2q^6t^4 + 2q^6t^3 \\ &\quad + q^4t^5 + 2q^2t^5 + t^6)/q^8t^3) \\ \mathrm{KR}'_3 &= (q^{12} + q^{10} + q^8t^2 + 3q^6t^2 + q^4t^2 + q^2t^4 + t^4)/(q^6t^2) \\ \mathrm{KR}_4 &= (q^{20} + 2q^{18}t + 2q^{14}t^3 + q^{14}t + 3q^{12}t^3 + 2q^{12}t^2 + 6q^{10}t^3 \\ &\quad + 2q^8t^4 + 3q^8t^3 + q^6t^5 + 2q^6t^3 + 2q^2t^5 + t^6)/(q^{10}t^3) \end{split}$$