Hyperconifold Singularities and Transitions

Rhys Davies

Mathematical Institute,
University of Oxford

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Based on **arXiv: 0911.0708, 1102.1428, 1103.3156**

Outline

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Hyperconifold singularities

Multiply-connected Calabi–Yau manifolds come from free holomorphic group actions: $X = \widetilde{X}/\mathbb{Z}_N$. Often a codimension-one family $\widetilde{X}_0$ has a fixed point p_0. In **arXiv:0911.0708**, I show that $\widetilde{X}_0$ is singular at p_0, typically having a node (conifold).

So the quotient space X_0 develops a singularity which is a $\mathbb{Z}_N$ quotient of the conifold, fixing only the singular point — a $\mathbb{Z}_N$-*hyperconifold*.

These singularities have local toric descriptions, just like the conifold. Furthermore, X_0 always admits a crepant, projective resolution, giving rise to *hyperconifold transitions*.

Hyperconifold transitions

To pass to X_0, we fix one complex structure parameter, and the resolved manifold $\widehat{X}$ has $N-1$ new divisor classes, so the change in the Hodge numbers is

$$\Delta(h^{11}, h^{21}) = (N-1, -1) \ .$$

Furthermore, $\widehat{X}$ is simply connected, so such transitions, unlike conifold transitions, change the fundamental group.

Important for Reid's fantasy — between them, conifold and hyperconifold transitions (along with flops) can change all topological data.

General quotient groups

More complicated groups can have free actions on Calabi–Yau threefolds.

- Typically, a proper subgroup $H < G$ develops a fixed point.
- The local geometry is just the H-hyperconifold.
- The fundamental group of a resolution is G/H^G.
  (H^G is the normal closure of H inside G.)

See RD, **arXiv:1103.3156** for details, and use of this to construct the first known Calabi-Yau threefold with fundamental group S_3, the symmetric group on three letters.

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Example — the $\mathbb{Z}_3$ quotient of the bicubic

$$\widetilde{X}^{2,83} \;=\; \begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}3\\3\end{bmatrix}$$

Take homogeneous coordinates $z_i, w_j\ ,\ i,j = 0,1,2$ on $\mathbb{P}^2\times\mathbb{P}^2$.

Define a $\mathbb{Z}_3$ action by

$$g : z_i \to \zeta^i z_i\ ,\ w_j \to \zeta^j w_j \quad ; \quad \zeta = e^{2\pi \mathrm{i}/3}\ .$$

A generic invariant bicubic hypersurface is smooth, and misses fixed points:

$$\alpha_0\, z_0^3 w_0^3 + z_0^2 w_0^2 (z_1 w_2 - z_2 w_1) + \ldots = 0\ .$$

We obtain a smooth quotient family $X^{2,29}$, with fundamental group $\mathbb{Z}_3$.

The $\mathbb{Z}_3$ quotient of the bicubic — creating a fixed point

$$\widetilde{X}^{2,83} = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \alpha_0\, z_0^3 w_0^3 + z_0^2 w_0^2 (z_1 w_2 - z_2 w_1) + \ldots = 0$$

One $\mathbb{Z}_3$-fixed point in the ambient space is given by $z_i = w_i = \delta_{i0}$. This will lie on the hypersurface $\widetilde{X}$ iff $\alpha_0 = 0$. Note that this is one condition on the complex structure.

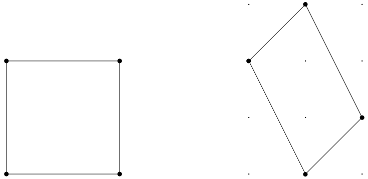
In local coords, with $z_0 \to 1, w_0 \to 1$, we have simply

$$\alpha_0 + z_1 w_2 - z_2 w_1 + \ldots = 0 \ ,$$

so for $\alpha_0 = 0$, the local geometry at the fixed point is just the conifold.

The $\mathbb{Z}_3$ quotient of the bicubic — the $\mathbb{Z}_3$-hyperconifold

So for $\alpha_0 = 0$, the quotient family $X^{2,29}$ develops a singularity which is locally a $\mathbb{Z}_3$ quotient of the conifold. This is also toric:



The toric diagrams for the conifold and the $\mathbb{Z}_3$-hyperconifold. The latter is obtained from the former by a lattice sub-division (and coordinate change).

The $\mathbb{Z}_3$ quotient of the bicubic — the resolution

The singular members of $X^{2,29}$ admit projective, crepant resolutions, given locally by a toric triangulation:

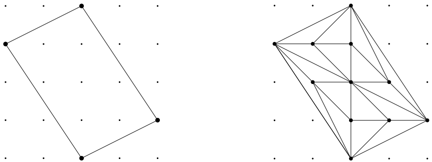
This introduces two new divisor classes, so $\Delta h^{1,1} = 2$, and since we fixed one complex structure parameter, $\Delta h^{2,1} = -1$. So the resolved Calabi–Yau family is $X^{4,28}$, and has $\pi_1 = \mathbf{1}$.

Known $\mathbb{Z}_N$-hyperconifolds

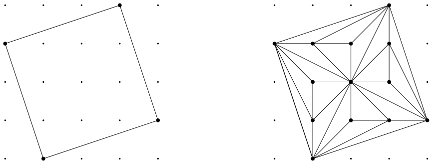
Free quotients of CICYs by $\mathbb{Z}_N$ exist for $N = 2, 3, 4, 5, 6, 8, 10, 12$.
(Braun, **arXiv:1003.3235**)

Corresponding $\mathbb{Z}_N$-hyperconifolds occur just as in the bicubic case. All have local toric descriptions, and all admit global projective, crepant resolutions.

So hyperconifold transitions to another Calabi-Yau always exist — there are no homological conditions to check, in contrast to the conifold case.



The $\mathbb{Z}_8$-hyperconifold and its resolution.



The $\mathbb{Z}_{10}$-hyperconifold and its resolution.

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Using the 'local mirror symmetry' formalism, we can study the mirrors to hyperconifolds.[1]

The mirror to a $\mathbb{Z}_N$-hyperconifold is a local Calabi–Yau with N nodes. Resolving the hyperconifold is mirror to smoothing these nodes, and vice-versa.

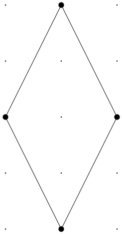
$$\mathbb{Z}_N\textit{-hyperconifold transition} \overset{\text{Mirror}}{\longleftrightarrow} N\textit{-nodal conifold transition.}$$

One can view this as a counter-example to an old conjecture of Morrison that the mirror to a conifold transition is a conifold transition.

[1] Thanks to Mark Gross. For some global examples, see **arXiv:1102.1428.**

Example: $\mathbb{Z}_4$-hyperconifold and its mirror

Toric diagram:



If we take coordinates (u, v, x, y) on $\mathbb{C}^2 \times \left(\mathbb{C}^*\right)^2$, then the mirror is

$$uv = 1 + x^{-1}y^2 + xy^2 + y^4 = (1 + x^{-1}y^2)(1 + xy^2)$$

This has nodes where each factor vanishes, i.e. where $x = -y^2$ and $y^4 = 1$. So indeed, it is four-nodal.

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Type IIB on a hyperconifold

Very similar to conifold, but important differences:

- Vanishing cycle is now $S^3/\mathbb{Z}_N$, not S^3.

  A $D3$-brane wrapping this has N ground states (choice of Wilson line).

- Hence N massless hypermultiplets at singular point, all of same charge.

  So $N-1$ hypermultiplets are flat directions $\leftrightarrow$ new Kähler moduli.

Full details in **arXiv:1102.1428.**

Conclusion

Hyperconifold singularities:

- Arise naturally in limits of multiply-connected compact manifolds.
- Admit projective crepant resolutions $\Rightarrow$ hyperconifold *transitions*.
  This allows construction of many new Calabi–Yau manifolds.
- Have a non-singular type II description, à la Strominger.
- Are mirror to varieties containing multiple ordinary nodes.
  (Hyperconifold transitions are mirror to ordinary conifold transitions.)

In principle, conifold transitions and hyperconifold transitions together might connect all compact Calabi-Yau threefolds.