A Generalized Borcea-Voisin Construction

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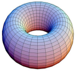
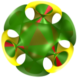
Question

Can we similarly generalize the construction of Borcea and Voisin?

Borcea-Voisin threefolds

Ingredients =  +  + Symmetry

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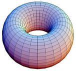
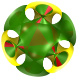
Ingredients $=$  $+$  $+$ Symmetry

- a torus E with a involution i (FREE)

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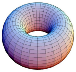
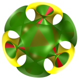
Ingredients $=$  $+$  $+$ Symmetry

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- a K3 surface S with a non-symplectic involution j, i.e. $j^*|_{H^{2,0}(S)} \equiv -\mathrm{id}$

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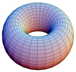
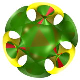
Ingredients =  +  + Symmetry

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Definition

The Borcea-Voising threefold associated to the above objects is

$$X = \frac{\widetilde{E \times X}}{\langle (i, j) \rangle}$$

Generalized Borcea-Voisin three- and fourfolds

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- $X_{i=1,2}$ a Calabi-Yau manifold endowed with a primitive non-symplectic automorphism j_i of order p, prime.
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There might not be a crepant resolution !!

Key tool: orbifold cohomology [Chen, W. & Ruan Y.]

$$\mathrm{H}^{*,*}_{\mathrm{orb}}(X/K) = \bigoplus_{g \in \mathrm{Conj}(K)} \bigoplus_{\Lambda \in \Phi(g)} H^{*-\kappa(g,\Lambda),*-\kappa(g,\Lambda)}(\Lambda)^K$$

Where $\Phi(g)$ is the set of irreducible components fixed by g, and $\kappa(g, \Lambda)$ is the age of g at a point of Λ.

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$$\mathrm{H}^{*,*}_{\mathrm{orb}}(X/K) = H^{*,*}(X)^K \oplus \bigoplus_{\Lambda\in\Phi(\gamma)} \bigoplus_{i=1}^{p-1} H^{*-\kappa(\gamma^i,\Lambda),*-\kappa(\gamma^i,\Lambda)}(\Lambda)$$

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- Moreover, action is essentially characterized by rank r of fixed locus Z on $H^2(S, \mathbb{Z})$ and by a where $\det Z = p^a$. In disguise, you have an action on the Gram graph.

Local picture ($p = 3$)

$$\begin{pmatrix} \zeta_3^2 & & \\ & \zeta_3^2 & \\ & & \zeta_3^2 \end{pmatrix}$$

$$\frac{1}{3}(2, 2, 2)$$

$$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}$$

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Point $\times$ Point $=$ Point

$$\begin{pmatrix} 1 & & \\ & \zeta_3 & \\ & & \zeta_3^2 \end{pmatrix}$$

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Curve $\times$ Point $=$ Curve

Earlier Work

- $p = 3$, $\dim X = 3$ [Rohde, J.C.] ✓

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- $p = 4$ $\dim X = 3$ [Garbagnati. A] partial
- $p = 2$, $\dim X = 4$ [Borcea, C.; Abe, M. and Sato, M.] partial
- $p > 2$, $\dim X = 4$ [Cynk, S. and Hulek, K.] partial

Results

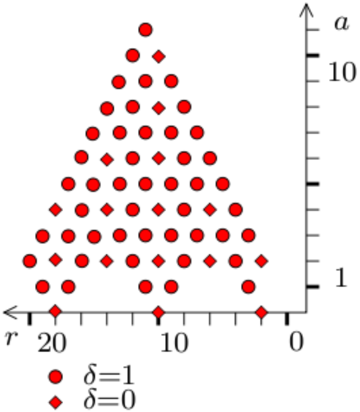
dim	p	#	Euler characteristic (χ)	Minimal $\lvert\chi\rvert$
3	3		$-62 + 12r$	-38
4	2		$888 - 60r_2 - 60r_1 + 6r_1r_2$	$-6, 0, 18$
	3	299	$408 - 36r_2 - 36r_1 + 6r_1r_2$	$-48, 0, 24$
	5	28	$174 - 21r_2 - 21r_1 + \frac{15}{2}r_1r_2$	24
	7	15	$\frac{304}{3} - \frac{40}{3}r_2 - \frac{40}{3}r_1 + \frac{28}{3}r_1r_2$	144
	11	6	$\frac{264}{5} - \frac{12}{5}r_2 - \frac{12}{5}r_1 + \frac{66}{5}r_1r_2$	96
	13	1	2184	2184
	17	1	1376	1376
	19	1	936	936

Mirror Symmetry

Question

Is there a mirror pairing within the realm of our families?

K3 lattices when $p = 2$



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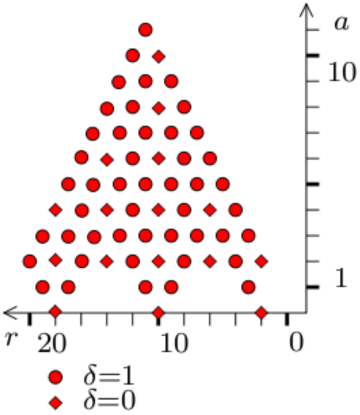
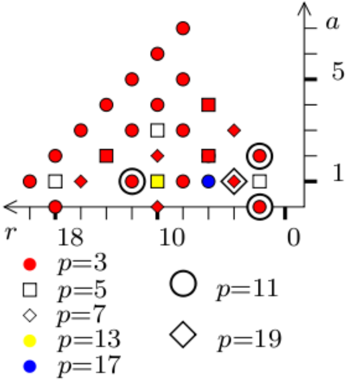


$r \leftrightarrow 20 - r$

K3 lattices when $p \geq 3$



Do our varieties come in pairs?

	$p = 2$	$p > 2$
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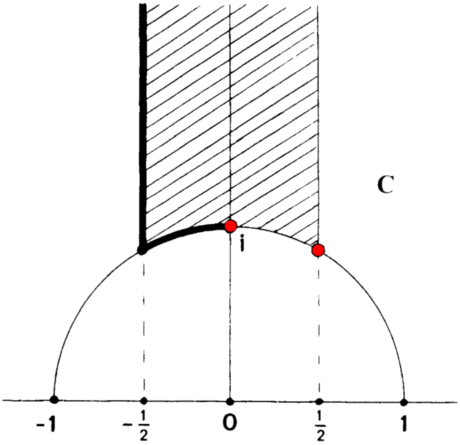

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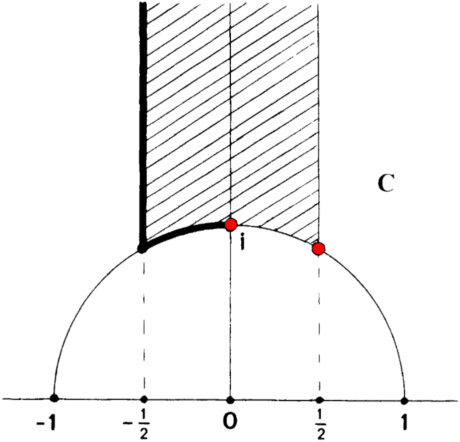

Also, Picard-Fuchs equation highlights lack of solutions with maximally unipotent monodromy.

In dimension 4

Problem with singularities

[Batyrev, V. & Dais, D.; Reid, M.] If X has a fixed point of type $\frac{1}{p}(2, p-1, 1, p-2)$ then X does not admit a crepant resolution.

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- For $p > 2$ all actions have fixed points of the above type $\rightarrow$ no resolution...

The future

What's next?

How do we build the mirrors of these generalized Borcea-Voisin varieties? Suggestions?

- Toric Geometry
- LG-model

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Thank you