Remarks on Fully Extended 3-Dimensional Topological Field Theories

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Work in progress with Constantin Teleman

Manifolds and Algebra: Abelian Groups

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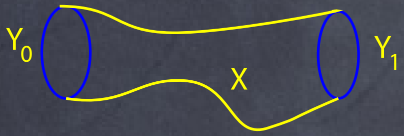

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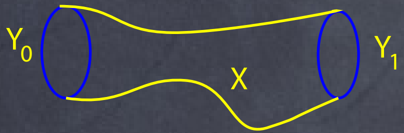

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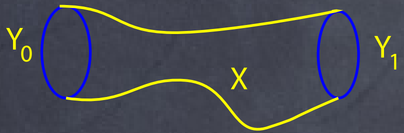

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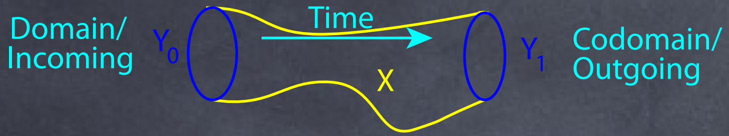
Many applications in homotopy theory: (i) framed bordism groups are stable homotopy groups of spheres (Pontrjagin-Thom construction); (ii) complex cobordism is universal among certain cohomology theories (Quillen)

Manifolds and Algebra: Symmetric Monoidal Categories

A more elaborate algebraic structure is obtained if we (i) do not identify bordant manifolds and (ii) remember the bordisms. So fix n and introduce a bordism category Bo_n whose objects are closed $(n-1)$-manifolds and morphisms are compact n-manifolds $X : Y_0 \to Y_1$



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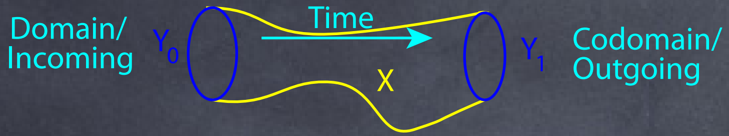

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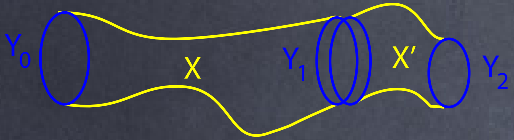

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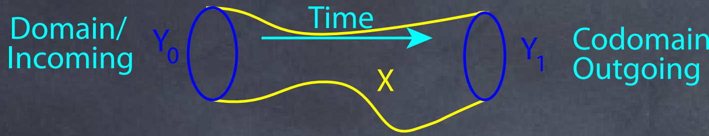

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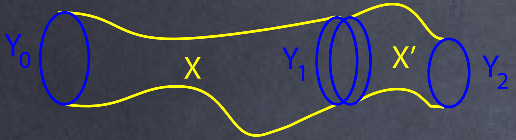

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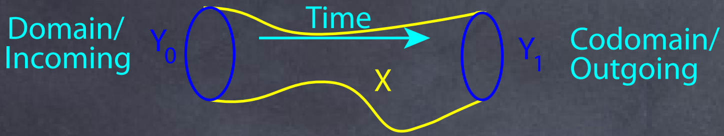

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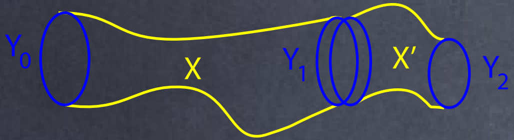

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Recover the abelian group Ω_{n-1} by declaring all morphisms to be invertible=*iso*morphisms. New information from non-invertibility.

Topological Quantum Field Theory

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E_n-algebra structure on $S^{n-1} \in \mathrm{Bo}_n$ via the generalized "pair of pants":

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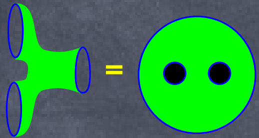

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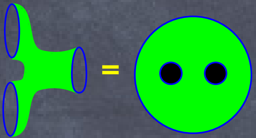

Therefore, $F(S^{n-1}) \in \mathbf{Vect}_{\mathbb{C}}$ is also an E_n-algebra (OPE)

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α factors through MT spectrum $\Rightarrow$ homotopy theory techniques

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Partial results in special cases (F., Walker, F.-Hopkins-Lurie-Teleman, Kapustin-Saulina, Bartels-Douglas-Henriques).

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A new algebraic gadget: the bordism (∞, n)-category Bord_n.

Bo_n: $(n-1)$-manifolds and n-manifolds with boundary

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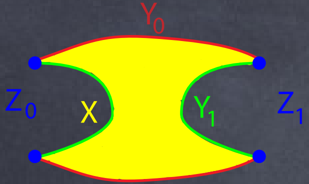

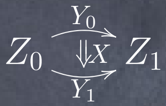

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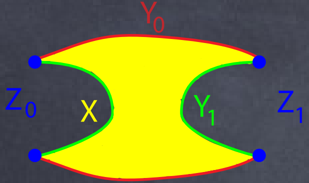

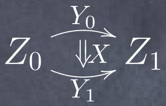

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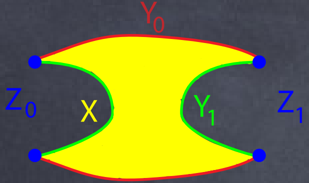

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For example, if $n = 3$ then typically $F(S^1)$ is a $\mathbb{C}$-linear category, also an E_2-algebra. $E_2(\mathbf{Cat}_{\mathbb{C}}) = \beta \otimes \mathbf{Cat}_{\mathbb{C}}$ are braided tensor categories.

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Full dualizable is a finiteness condition. For example, in a TQFT the vector spaces attached to closed $(n-1)$-manifolds are finite dimensional. In an extended theory $F(\mathrm{pt})$ satisfies analogous finiteness conditions.

Spheres and Invertibility

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Remark 3: As I explain later we apply this to $n = 4$, $k = 2$, and $\mathcal{C} = \beta\!\otimes\!\mathbf{Cat}_{\mathbb{C}}$ the symmetric monoidal 4-category of braided tensor categories. Then α is the anomaly theory for Chern-Simons, and we construct Chern-Simons as a 0-1-2-3 *anomalous* theory.

Proof Sketch

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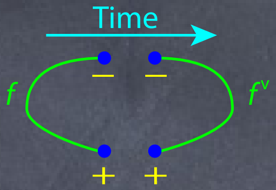

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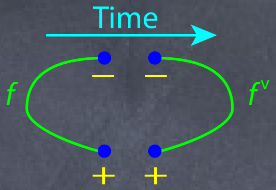
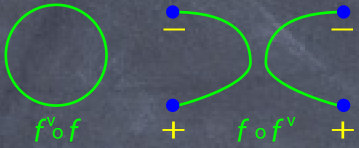
$$f^\vee = D^1 : S^0 \longrightarrow 1$$



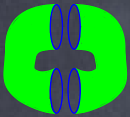
We arrive at a statement about 1-manifolds: the compositions

$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$



are the identity.

Let's now consider $n = 2$ where we assume that S^1 is invertible. We apply an easy algebraic lemma which asserts that invertible objects are dualizable and the dualization data is invertible. For S^1 these data are dual cylinders, and so the composition $S^1 \times S^1$ is also invertible.



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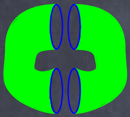

Lemma: Suppose $\mathcal{D}$ is a symmetric monoidal category, $x \in \mathcal{D}$ is invertible, and $g\colon 1 \to x$ and $h\colon x \to 1$ satisfy $h \circ g = \mathrm{id}_1$. Then $g \circ h = \mathrm{id}_x$ and so each of g, h is an isomorphism.

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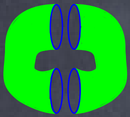

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Proof sketch: x^{-1} is a dual of x, $g^\vee = x^{-1}g\colon x^{-1} \to 1$, $h^\vee = x^{-1}h\colon 1 \to x^{-1}$, so the lemma follows from $(h \circ g)^\vee = \mathrm{id}_1$.

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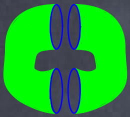

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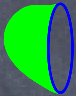
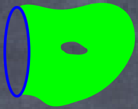
Apply the lemma to the 2-morphisms



$$g = D^2\colon 1 \longrightarrow S^1$$

$$h = S^1 \times S^1 \backslash D^2\colon S^1 \longrightarrow 1$$



Conclude that $S^1 \cong 1$ and $S^2 = g^\vee \circ g$ is invertible.
Also, $g \circ g^\vee = \mathrm{id}_{S^1} \otimes S^2$, a simple surgery.

Recall that we must prove that the compositions

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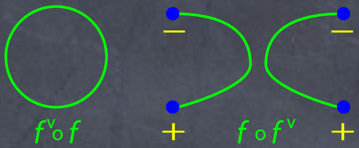
are the identity. We just did the first.



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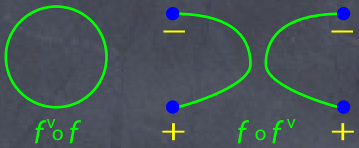

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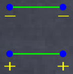
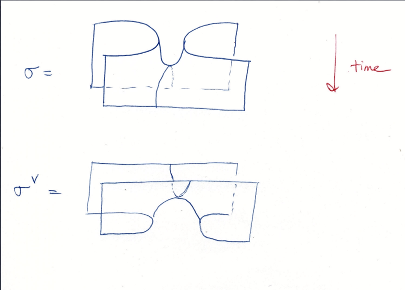
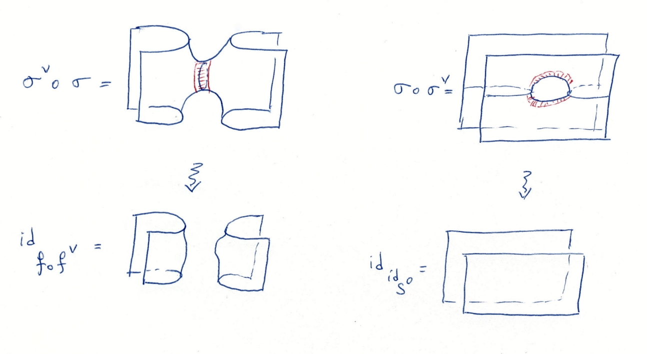
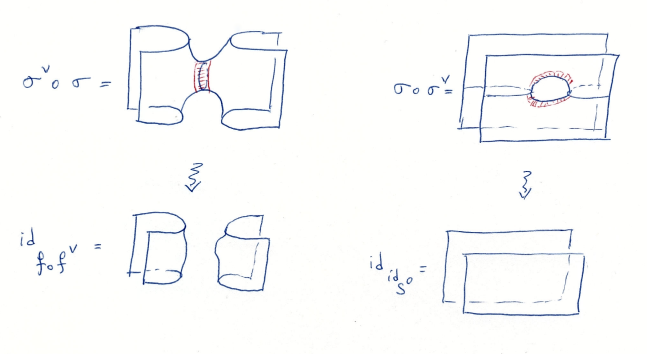
For the second the identity is  and we will show that the saddle $\sigma\colon f \circ f^\vee \to \mathrm{id}_{S^0}$ is an isomorphism with inverse $\sigma^\vee \otimes S^2$.



The saddle σ is diffeomorphic to $D^1 \times D^1$, which is a manifold with corners. Its dual $\sigma^\vee$ is the time-reversed bordism.



Inside each composition $\sigma^\vee \circ \sigma$ and $\sigma \circ \sigma^\vee$ we find a cylinder $\mathrm{id}_{S^1} = D^1 \times S^1$, which is $(S^2)^{-1} \otimes g \circ g^\vee = (S^2)^{-1} \otimes (S^0 \times D^2)$ by a previous argument. Making the replacement we get the desired isomorphisms to identity maps.



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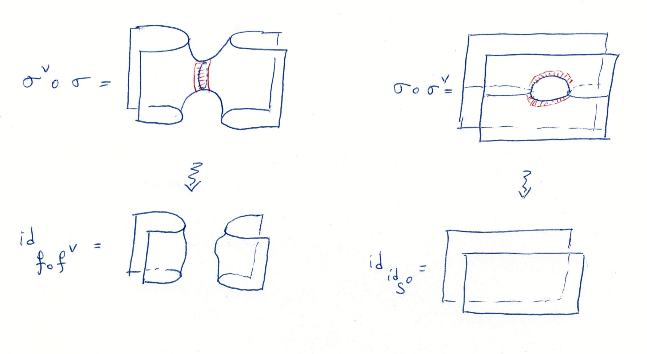

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In higher dimensions we see a kind of Poincaré duality phenomenon: we prove invertibility by assuming it in the middle dimension. A new ingredient—a dimensional reduction argument—also appears.

Application to Modular Tensor Categories

Let F denote the usual quantum Chern-Simons 1-2-3 theory for some gauge group G. It was introduced by Witten and constructed by Reshetikhin-Turaev from quantum group data. The latter construction works for any modular tensor category A, a braided tensor category which satisfies finiteness conditions (semisimple with finitely many simples, duality, etc.) and a nondegeneracy condition (the S matrix is invertible). Then F is a 1-2-3 theory with $F(S^1) = A$.

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Let A be a braided tensor category with braiding $\beta(x, y) \colon x \otimes y \to y \otimes x$.

Theorem: The nondegeneracy condition on A is equivalent to

$$\{x \in A : \beta(y, x) \circ \beta(x, y) = \mathrm{id}_{x \otimes y} \text{ for all } y \in A\} = \{\text{multiples of } 1 \in A\}.$$

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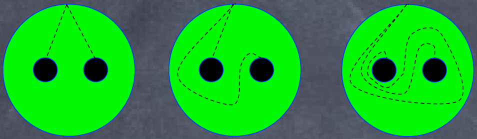
Recall that $\beta\otimes\mathbf{Cat}_{\mathbb{C}} = E_2(\mathbf{Cat}_{\mathbb{C}})$.



Braided tensor categories form the objects of a 4-category!

object	category number
element of $\mathbb{C}$	-1
$\mathbb{C}$-vector space	0
$\mathbf{Vect}_{\mathbb{C}}$	1
$\mathbf{Cat}_{\mathbb{C}}$	2
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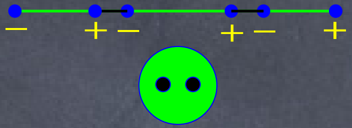

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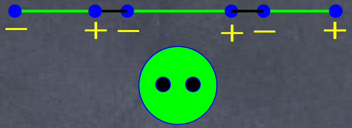

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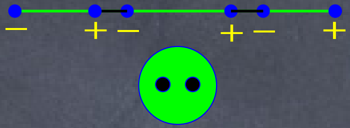

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So, given sufficient finiteness, a braided tensor category determines (using the cobordism hypothesis) an extended 4-dimensional TQFT

$$\alpha\colon \mathrm{Bord}_4 \to \beta\otimes\mathbf{Cat}_{\mathbb{C}}$$

In the theory α we compute

$$\alpha(S^2) = \{x \in A : \beta(y,x) \circ \beta(x,y) = \mathrm{id}_{x \otimes y} \text{ for all } y \in A\} \in \mathbf{Cat}_{\mathbb{C}}$$

Recall that for a modular tensor category this “higher center” of A is the tensor unit $1 = \mathbf{Vect}_{\mathbb{C}}$, which in particular is invertible.

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We believe that this is the anomaly theory for a 0-1-2-3 extension of the 1-2-3 theory F with $F(S^1) = A$. In the remainder of the lecture I will explain this idea.

Anomalous Field Theories

The top-level values of an n-dimensional field theory $F\colon \mathrm{Bo}_n \to \mathbf{Vect}_{\mathbb{C}}$ are complex numbers $F(X^n) \in \mathbb{C}$, the *partition function* of a closed n-manifold. In an anomalous field theory f there is a complex line L_X associated to X and the partition function $f(X) \in L_X$ lies in that line.

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f is an n-dimensional theory with values in the $(n+1)$-dimensional theory α. We write $f\colon 1 \to \alpha$ in the sense that $f(X)\colon 1 \to \alpha(X)$ for all X. (1 is the trivial theory.) If α is invertible we say f is anomalous with anomaly α. The same ideas apply in *extended* field theories.

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Remark: The notion of α-valued field theory makes sense even if α is not invertible and also for non-topological field theories. Examples: (i) $n = 2$ chiral WZW valued in topological Chern-Simons, (ii) $n = 6$ (0,2)-(super)conformal field theory valued in a 7-dimensional theory.

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Remark: This is a specialization of the notion of a domain wall.

Fully Extended Chern-Simons

Recall that a modular tensor category A determines an invertible extended field theory $\alpha\colon \mathrm{Bord}_4 \to \beta^{\otimes}\mathbf{Cat}_{\mathbb{C}}$ with values in the 4-category of braided tensor categories, or equivalently E_2-algebras in $\mathbf{Cat}_{\mathbb{C}}$.

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For example, the composition $1 \xrightarrow{f(S^1)} \alpha(S^1) \xrightarrow{\alpha(D^2)} 1$ is $F(S^1) = A$, where the bordism $D^2\colon S^1 \to 1$ is used to trivialize the anomaly on S^1.