Deformations of G_2 structures

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1. Motivation
2. Basics of the group G_2
3. Overview of G_2 structures
4. Differential forms on G_2 structure manifolds
5. G_2 structure torsion
6. Deformations of G_2 structures
7. Concluding remarks

1. Motivation

	String theory	M-theory	Math description
No fluxes + SUSY	Calabi-Yau	G_2 holonomy	$\nabla\xi = 0$, Ricci flat
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Note: equations of motion coming from leading terms of 11D supergravity require fluxes to vanish if the 7-manifold is compact, so to allow non-zero fluxes either need a non-compact manifold (e.g. hep-th/0010282 Becker and Becker), or higher-order corrections to the action, or sources

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Note: Generally, we will denote the Hodge dual of φ_0 by ψ

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Note: if a 3-form defines a non-degenerate bilinear form this way, then necessarily, it will define either a G_2 structure with positive definite metric or a *split* G_2 structure, with signature (4,3)

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- For *3*-forms, have

$$\Lambda^3_1 = \{a\varphi : a \text{ a function}\}$$
$$\Lambda^3_7 = \{v \lrcorner * \varphi : \ v \text{ a vector field}\}$$
$$\Lambda^3_{27} = \left\{h^d{}_{[a}\varphi_{|d|bc]} : h_{ab} \text{ traceless, symmetric}\right\}$$

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- From this, get

$$\boxed{T_{ab} = \frac{1}{24} \left(\nabla_a \varphi_{bcd} \right) \psi_m{}^{bcd}}$$

Torsion decomposition

- Can split T_{ab} according to representations of G_2:

$$T = \tau_1 g + \tau_7 \lrcorner \varphi + \tau_{14} + \tau_{27}$$

τ_1	function	1-dim rep	W_1
τ_7	1-form	7-dim rep	W_7
τ_{14}	2-form in Λ^2_{14}	14-dim rep	W_{14}
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- Determine $d\varphi$ and $d\psi$:

$$d\varphi = 4\tau_1 \psi - 3\tau_7 \wedge \varphi - *\tau_{27}$$
$$d\psi = -4\tau_7 \wedge \psi - 2 * \tau_{14}$$

Properties of torsion components

- If all torsion components vanish, then we have

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- In some specific cases have the following constraints:

Torsion class	Constraint
W_1	$d\tau_1 = 0$
W_7	$d\tau_7 = 0$
W_{14}	$d^*\tau_{14} = 0$
$W_1 \oplus W_7$	$d\tau_1 = \tau_1\tau_7$
$W_1 \oplus W_{14}$	$\tau_1\tau_{14} = 0$

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- Alternatively, can consider only infinitesimal behaviour, e.g. Karigiannis math/0702077, SG & Yau 0802.0723, SG 0911.2185

Properties of general deformations

- Under a general deformation, $\varphi \longrightarrow \tilde{\varphi} = \varphi + \chi$, it can be shown that the new metric becomes

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$$s_{ab} = g_{ab} + \frac{1}{2}\chi_{mn(a}\varphi_{b)}{}^{mn} + \frac{1}{8}\chi_{amn}\chi_{bpq}\psi^{mnpq} + \frac{1}{24}\chi_{amn}\chi_{bpq}\left(*\chi\right)^{mnpq}$$

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- Moreover, can show that for the new metric, $\tilde{*}\tilde{\varphi} = \tilde{\psi}$ is given by

$$\tilde{\psi}_{abcd} = \left(\frac{\det g}{\det \tilde{g}}\right)^{\frac{5}{2}} \left(\psi^{mnpq} + *\chi^{mnpq}\right) s_{ma}s_{nb}s_{pc}s_{qd}$$

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- The Levi-Civita connection of the new metric differs by

$$\delta\Gamma_{a\ \ c}^{\ b} = \frac{1}{2}\left(\frac{\det g}{\det \tilde{g}}\right)^{\frac{1}{2}} \left(\tilde{g}^{\tilde{b}\tilde{d}}\left(\nabla_c s_{ad} + \nabla_a s_{cd} - \nabla_d s_{ac}\right) - \frac{1}{9}\left(\delta_a^b \delta_c^e + \delta_c^b \delta_a^e - \tilde{g}_{ac}\tilde{g}^{\tilde{b}\tilde{e}}\right)\tilde{g}^{\tilde{m}\tilde{n}}\nabla_e s_{mn}\right)$$

Deformed torsion

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$$\tilde{T}_{an} = \frac{1}{24}\left(\frac{\det g}{\det \tilde{g}}\right)\left(\left(24T_a{}^m + T_a{}^e\psi_{ebcd} * \chi^{mbcd} + \psi^{mbcd}\nabla_a\chi_{bcd} + \nabla_a\chi_{bcd} * \chi^{mbcd}\right)s_{mn}\right.$$
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$$\left(\tilde{\tau}_{14}\right)_{an} = \frac{2}{3}\tilde{T}_{[an]} - \frac{1}{6}\tilde{T}_{mp}\tilde{\psi}^{\tilde{m}\tilde{p}}{}_{an} \qquad \left(\tilde{\tau}_{27}\right)_{an} = \tilde{T}_{(an)} - \tau_1\tilde{g}_{an}$$

Conformal deformations

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- So this deformation only affects the W_7 torsion component
- Recall that in $W_1 \oplus W_7$ class, if τ_1 is non-vanishing, then

$$\tau_7 = \frac{1}{\tau_1} d\tau_1$$

Hence taking $f = \tau_1$ will reduce the torsion class to W_1

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Deformations in Λ_7^3

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- For convenience, denote $|v|^2 = M$. Then we have:

$$\tilde{g}_{ab} = (1+M)^{-\frac{2}{3}} \left(g_{ab}(1+M) - v_a v_b\right)$$
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$$\begin{aligned}
\tilde{T}_{an} = {} & (1+M)^{-\frac{4}{3}} \Big(v_1 \left(v_a v_n - (1+M) g_{an} \right) - \tfrac{4}{3} (1+M) v_1 \quad \varphi_{anm} v^m - \left(1 + \tfrac{4}{3} M\right) \varphi_{anm} v_7^m \\
& - \tfrac{1}{3} \psi_{anmp} v^m v_7^p + \tfrac{5}{3} v_a \varphi_{nmp} \ v^m v_7^p + \tfrac{4}{3} v_n \varphi_{amp} \ v^m v_7^p + \tfrac{1}{3} v_7^m v_m \varphi^p{}_{an} v_p + \tfrac{1}{3} v_n (v_7)_a \\
& + \tfrac{8}{3} v_a (v_7)_n - (1+M) (v_{14})_{an} - 2 v_m (v_{14})^m{}_{[a} v_{n]} + \tfrac{1}{3} \varphi_{anm} v_{14}^{mp} v_p + \tfrac{1}{3} \psi_{anmp} v_q v^m v_{14}^{pq} \\
& - (1+M) (v_{27})_{an} + v_m (v_{27})^m{}_a v_n - (1+M) \varphi^{mp}{}_a (v_{27})_{pn} v_m - \tfrac{1}{3} \varphi_{anm} v_{27}^{mp} v_p \\
& + \tfrac{1}{3} \psi_{anmp} v^m v_{27}^{pq} v_q + v_a \varphi_{nmp} v^m v_{27}^{pq} v_q - \tfrac{1}{3} \varphi_{an}{}^m v_m v_{27}^{pq} v_p v_q \Big) \\
& + (1+M)^{-\frac{1}{3}} \Big(\tau_1 g_{an} + \tau_1 \varphi^m{}_{an} v_m + \varphi_{anm} \tau_7^m + v_a (\tau_7)_n - g_{an} \tau_7^m v_m + \psi_{anmp} \tau_7^m v^p \\
& + (\tau_{14})_{an} - \varphi_{nmp} v^m (\tau_{14})^p{}_a + (\tau_{27})_{an} + \varphi_{nmp} v^m (\tau_{27})^p{}_a \Big)
\end{aligned}$$

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- **Theorem 3** There exist no deformations from the strict torsion class W_1 to itself.
- **Theorem 4** A deformation from the strict class $W_1 \oplus W_7$ to the strict class W_1 exists if and only if the original metric is a warped product with an interval (with a particular warp factor)

7. Concluding remarks

- Even for a relatively simple deformation in Λ^3_7, the torsion deforms in a very complicated fashion. It would however be interesting to investigate what happens to the 14- and 27-dimensional torsion components
- Much more difficult would be to perform the same analysis for deformations in Λ^3_{27}.
- However in the case of vanishing torsion this could eventually give some answers whether or not the moduli space of manifolds with G_2 holonomy is unobstructed, perhaps along the lines of the Bogomolov-Tian-Todorov theorem for Calabi-Yau moduli spaces.