Non-Commutative Solitons and Quasi-determinants.

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Based on

- **Claire R.Gilson (Glasgow), MH and Jonathan J.C.Nimmo (Glasgow), ``Backlund trfs and the Atiyah-Ward ansatz for NC anti-self-dual (ASD) Yang-Mills (YM) eqs.'' Proc. Roy. Soc. A465 (2009) 2613 [arXiv:0812.1222].**
- **MH**, **``Noncommutative Solitons and Quasideterminants (review)'' [arXiv:1101.0005]**

Today's NC equations

- **NC Anti-Self-Dual Yang-Mills (ASDYM) eq.**

in Yang's form

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$

- **NC KdV eq.**

Spell: All products are replaced with the Moyal products.

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u * u_x + u_x * u)$$

The Moyal product: (NC and associative)

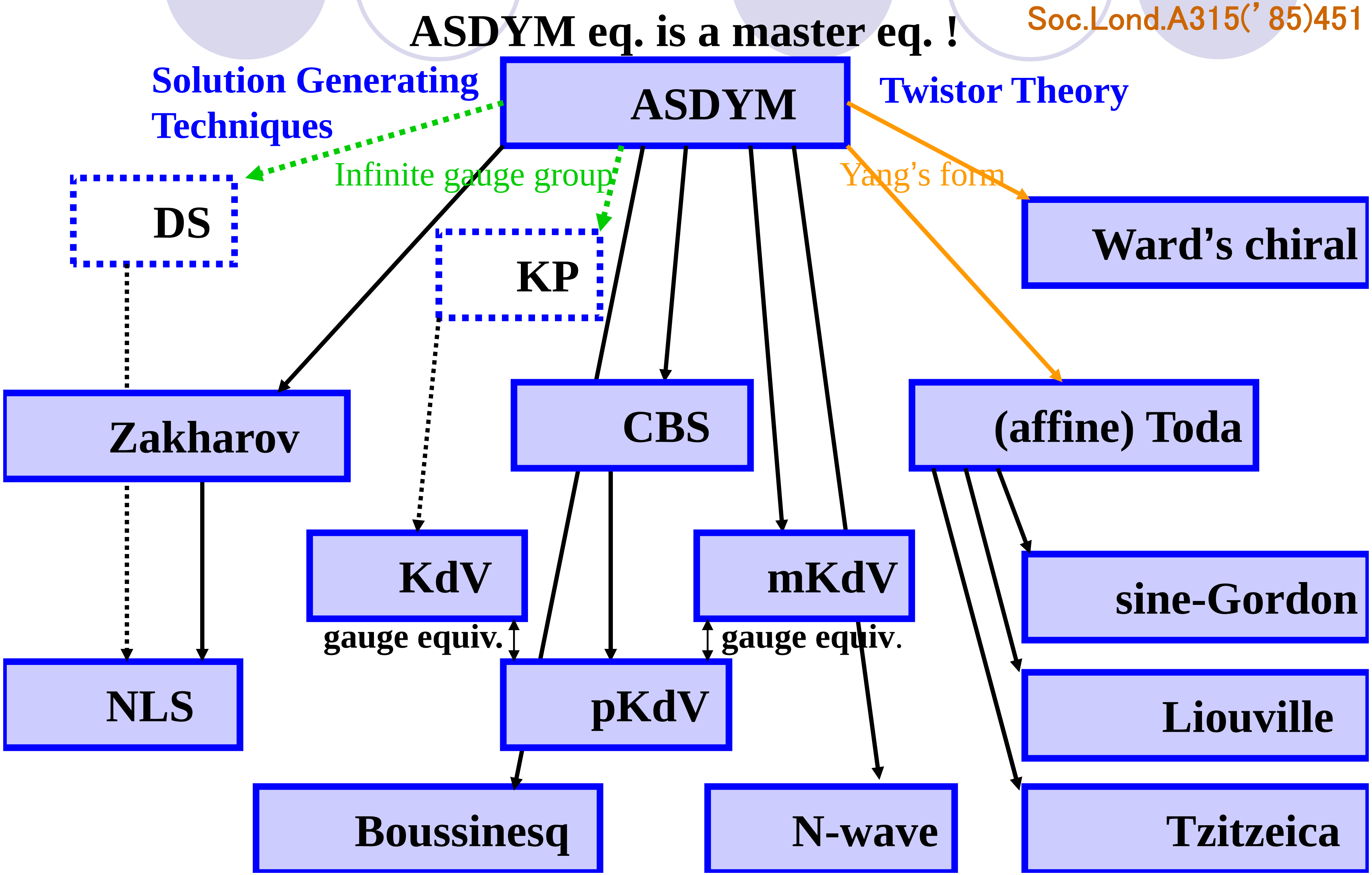
$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x) = f(x) g(x) + i \frac{\theta^{\mu\nu}}{2} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)$$

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

: the NC space!

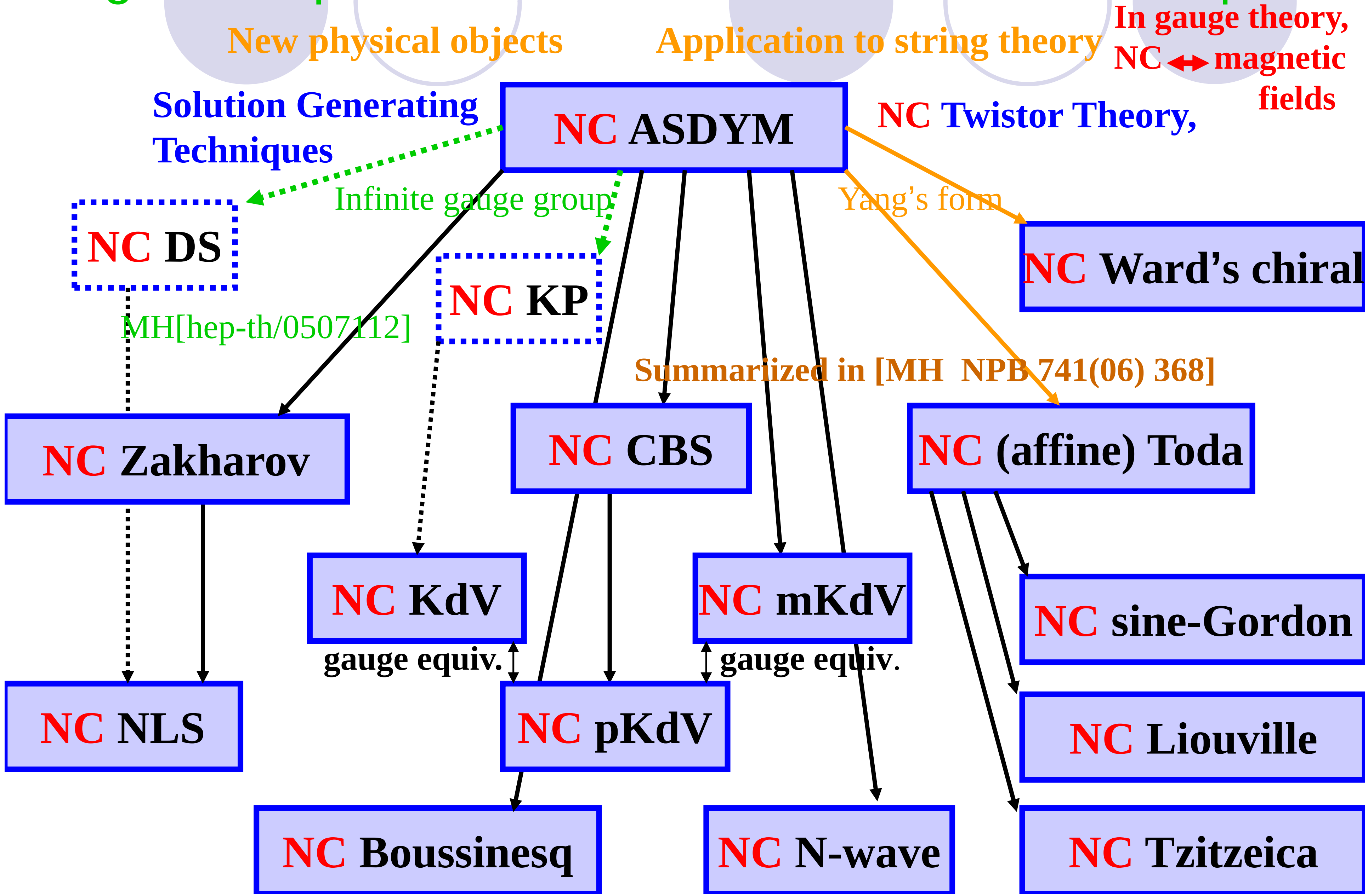
Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy. Soc.Lond.A315('85)451



NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

MH &K.Toda, PLA [hepth/0211148]



Reduction to NC KdV from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$(z, \tilde{z}, w, \tilde{w}) \to (t, x) = (z, w + \tilde{w})$$

Reduction conditions

$$A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_w = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, A_z = \begin{pmatrix} \frac{1}{2}q'' + q'q & -q' \\ \frac{1}{4}q''' + \frac{1}{2}(q'^2 + qq'' + q''q) + qq'q & -\frac{1}{2}q'' - qq' \end{pmatrix}$$

NOT traceless → U(1) part is crucial!

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'u + uu') \qquad u = 2q'$$

:NC KdV eq.!

The NC KdV eq. has integrable-like properties:

$$L=\partial_x+\frac{u}{2}\partial_x^{-1}+f(u^{(\prime)})\partial_x^{-2}+\cdots$$: a pseudo-diff. operator

- possesses infinite conserved densities:

$$\sigma_n=res_{-1}L^n+\frac{3}{4}\theta((res_{-1}L^n)\lozenge u''-2(res_{-2}L^n)\lozenge u')$$

MH, JMP46 (2005) [hep-th/0311206]

$res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x):=f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

$$[t,x]=i\theta$$

- has exact N-soliton solutions:

$$u=2\partial_x\sum_{i=1}^{N}(\partial_x W_i)W_i^{-1}$$

Etingof-Gelfand-Retakh, MRL [q-alg/9701008]
MH, JHEP [hep-th/0610006]
cf. Paniak, [hep-th/0105185]

$W_i:=\left|W(f_1,\dots,f_i)\right|_{i,i}$:quasi-determinant of Wronski matrix

$$f_i=\exp\xi(x,\alpha_i)+a_i\exp(-\xi(x,\alpha_i))\qquad \xi(x,\alpha)=x\alpha+t\alpha^3$$

Reduction to NC Tzitzeica eq.

- **Start with NC Yang's eq.**

$$\partial_z(J^{-1}\partial_{\tilde{z}}J)-\partial_w(J^{-1}\partial_{\tilde{w}}J)=0$$

- **(1) Take a special reduction condition:**

$$J=\exp(-E_-\tilde{w})g(z,\tilde{z})\exp(E_+w)$$

$$E_+=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

We get a reduced Yang's eq.

$$\partial_z(g^{-1}\partial_{\tilde{z}}g)-[E_-,g^{-1}E_+g]=0$$

$$E_-=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

- **(2) Take a further reduction condition:**

$$g=\exp(\rho)diag\,(\exp(\omega),\exp(-\omega),1)$$

We get (a set of) NC Tzitzeica eq.:

$$\partial_z(\exp(-\omega)\partial_{\tilde{z}}\exp(\omega))+\partial_z(\exp(-\omega)V\exp(\omega))=\exp(\omega)-\exp(-2\omega),$$
$$\partial_z(\exp(\omega)\partial_{\tilde{z}}\exp(-\omega))+\partial_z(\exp(\omega)V\exp(-\omega))=\exp(-2\omega)-\exp(\omega),$$
$$\partial_zV=\partial_z(\exp(-\rho)\partial_{\tilde{z}}\exp(\rho))=0$$

$$\left(\xrightarrow{\theta\to 0}\omega_{z\tilde{z}}=\exp(\omega)-\exp(-2\omega)\right)$$

Plan of this talk

1. Introduction
2. Quasideterminants
3. Backlund Transforms for NC ASDYM eqs.
4. Conclusion and Discussion

2. Quasi-determinants

[For a survey, see Gelfand, Gelfand, Retakh, Wilson, math.QA/0208146]

- Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.
- [Def] For an n by n matrix $X=(x_{ij})$ and the inverse $Y=(y_{ij})$ of X, quasi-determinant of X is directly defined by

X^{ij}: the matrix obtained from X deleting i-th row and j-th column

$$|X|_{ij} = y_{ji}^{-1} \quad \left(\xrightarrow{\text{In commutative limit}} \frac{(-1)^{i+j}}{\det X^{ij}} \det X \right)$$

proportional to the determinant or ratio of determinants.

$$|X|_{ij} = \begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$$

: A convenient notation

Examples of quasi-determinants

$$n=1: \quad |X|_{ij} = x_{ij}$$

$$n=2: \quad |X|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12}\cdot x_{22}^{-1}\cdot x_{21}, \quad |X|_{12} = \begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$

$$|X|_{21} = \begin{vmatrix} x_{11} & x_{12} \\ \boxed{x_{21}} & x_{22} \end{vmatrix} = x_{21} - x_{22}\cdot x_{12}^{-1}\cdot x_{11}, \quad |X|_{22} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix} = x_{22} - x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3: \quad |X|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11} - (x_{12}, x_{13}) \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$$

$$= x_{11} - x_{12}\cdot(x_{22} - x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21} - x_{13}\cdot(x_{32} - x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$

$$- x_{12}\cdot(x_{23} - x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31} - x_{13}\cdot(x_{33} - x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$

$$\cdots$$

Note:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & \underline{-A^{-1}B(D-CA^{-1}B)^{-1}} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (A-BD^{-1}C)^{-1} & \underline{(C-DB^{-1}A)^{-1}} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

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Some identities of quasideterminants

- **Homological relation**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}$$

e.g.

$$C - DB^{-1}A = \begin{vmatrix} A & B \\ \boxed{C} & D \end{vmatrix} = \begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} \begin{vmatrix} A & B \\ \boxed{0} & 1 \end{vmatrix} = (D - CA^{-1}B)(-B^{-1}A)$$

The identity in the previous page!

Some identities of quasideterminants

- **NC Jacobi's identitiy**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

$``D - CA^{-1}B"$

$$\xrightarrow{\theta \to 0} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} |A| = \begin{vmatrix} A & C \\ E & i \end{vmatrix} \begin{vmatrix} A & B \\ D & f \end{vmatrix} - \begin{vmatrix} A & B \\ E & h \end{vmatrix} \begin{vmatrix} A & C \\ D & g \end{vmatrix}$$

:(commutative) Jacobi's id.

3. Backlund transform for NC ASDYM eqs.

- **In this section, we give Backlund transformations for the NC ASDYM equation.**
- **The proof is made very simply by using identities of quasideterminants.**
- **The generated solutions are represented in terms of quasideterminants, which contain not only finite-action solutions (NC instantons) but also infinite-action solutions (non-linear plane waves and so on.)**

Backlund trf. for NC ASDYM eq. G=GL(2)

- The J matrix can be reparametrized as follows

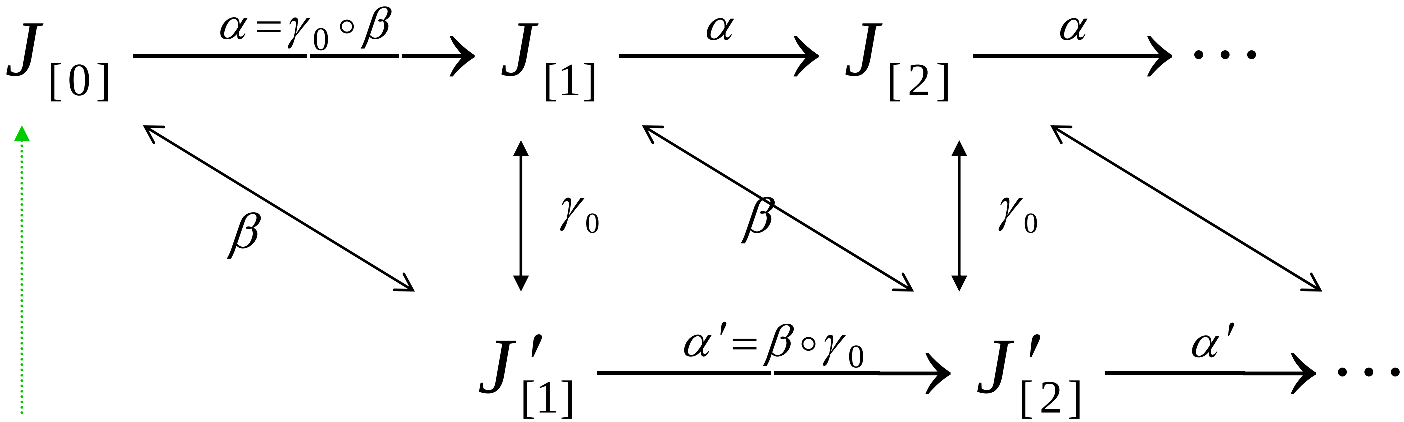
$$J=\begin{pmatrix} f-gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{pmatrix}$$

- Then the following two kind of trfs. leave NC ASDYM eq. $\partial_z(J^{-1}\partial_{\tilde{z}}J)-\partial_w(J^{-1}\partial_{\tilde{w}}J)=0$ as it is:

$$\beta:\begin{cases} \partial_z e^{new}=-f^{-1}g_{\tilde{w}}b^{-1},\ \partial_w e^{new}=-f^{-1}g_{\tilde{z}}b^{-1}, \\ \partial_{\tilde{z}}g^{new}=-b^{-1}e_w f^{-1},\ \partial_{\tilde{w}}g^{new}=-b^{-1}e_z f^{-1}, \\ f^{new}=b^{-1},\ b^{new}=f^{-1} \end{cases}$$

$$\gamma_0:\begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new} \end{pmatrix}=\begin{pmatrix} (b-ef^{-1}g)^{-1} & (g-fe^{-1}b)^{-1} \\ (e-bg^{-1}f)^{-1} & (f-gb^{-1}e)^{-1} \end{pmatrix}$$

- Both trfs. are involutive ($\beta \circ \beta = id, \gamma_0 \circ \gamma_0 = id$), but the combined trf. $\gamma_0 \circ \beta$ is non-trivial.)
- Then we could generate various (non-trivial) solutions of NC ASDYM eq. from a (trivial) seed solution (so called, NC Atiyah-Ward ansatz solutions)



A seed solution:

$$b_{[0]} = f_{[0]} = e_{[0]} = g_{[0]} = \Delta_0^{-1},$$

$$\partial^2 \Delta_0 = (\partial_z \partial_{\tilde{z}} - \partial_w \partial_{\tilde{w}}) \Delta_0 = 0$$

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

Explicit Atiyah-Ward ansatz solutions of NC ASDYM eq. G=GL(2)

[Gilson-MH-Nimmo, arXiv:0709.2069]

$$b_{[0]} = f_{[0]} = e_{[0]} = g_{[0]} = \Delta_0^{-1}, \quad \partial^2 \Delta_0 = 0$$

$$b_{[1]} = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \quad f_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \boxed{\Delta_0} \end{vmatrix}^{-1}, \quad e_{[1]} = \begin{vmatrix} \Delta_0 & \boxed{\Delta_{-1}} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \quad g_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \boxed{\Delta_1} & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = \partial_{\tilde{w}} \Delta_1, \quad \partial_z \Delta_{-1} = \partial_{\tilde{w}} \Delta_0, \quad \partial_w \Delta_0 = \partial_{\tilde{z}} \Delta_1, \quad \partial_w \Delta_{-1} = \partial_{\tilde{z}} \Delta_0$$

$$f_{[n]} = \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad b_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \boxed{\Delta_0} \end{vmatrix}^{-1},$$

$$e_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad g_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_0 \end{vmatrix}^{-1}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

Explicit Atiyah-Ward ansatz solutions of NC ASDYM eq. G=GL(2)

[Gilson-MH-Nimmo, arXiv:0709.2069]

$$f'_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \boxed{\Delta_0} \end{vmatrix}, \quad b'_{[n]} = \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix},$$

$$e'_{[n]} = \begin{vmatrix} \Delta_{-1} & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_{n-2} & \cdots & \Delta_{-1} \end{vmatrix}, \quad g'_{[n]} = \begin{vmatrix} \Delta_1 & \cdots & \Delta_{2-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_1 \end{vmatrix}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

$$\beta : \begin{cases} \partial_z e^{new} = -f^{-1} g_{\tilde{w}} b^{-1}, \ \partial_w e^{new} = -f^{-1} g_{\tilde{z}} b^{-1}, \\ \partial_{\tilde{z}} g^{new} = -b^{-1} e_w f^{-1}, \ \partial_{\tilde{w}} g^{new} = -b^{-1} e_z f^{-1}, \\ f^{new} = b^{-1}, \ b^{new} = f^{-1} \end{cases}$$

e.g. $b'_{[1]} = f'_{[1]} = \Delta_0, \ e'_{[1]} = \Delta_{-1}, \ g'_{[1]} = \Delta_1$

$$\partial_z \Delta_0 = \partial_{\tilde{w}} \Delta_1, \ \partial_z \Delta_{-1} = \partial_{\tilde{w}} \Delta_0, \ \partial_w \Delta_0 = \partial_{\tilde{z}} \Delta_1, \ \partial_w \Delta_{-1} = \partial_{\tilde{z}} \Delta_0$$

The proof is in fact very simple!

- Proof is made simply by using only special identities of quasideterminants. (NC Jacobi's identities and homological relations, Gilson-Nimmo's derivative formula etc.)
- In other words, ``NC Backlund trfs are identities of quasideterminants.''
  This is an analogue of the fact in lower-dim. commutative theory: ``Backlund trfs are identities of determinants.''

An example

- γ_0-**transformation**

$$b_{[n]}^{-1} = f'_{[n]} - g'_{[n]} {b'_{[n]}}^{-1} e'_{[n]}$$

$\Leftrightarrow$

$$\begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \boxed{\Delta_0} \end{vmatrix} =$$

$$\begin{vmatrix} \Delta_0 & \cdots & \Delta_{1-n} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \boxed{\Delta_0} \end{vmatrix} - \begin{vmatrix} \Delta_1 & \cdots & \Delta_{2-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_1 \end{vmatrix} \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{1-n} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \Delta_0 \end{vmatrix}^{-1} \begin{vmatrix} \Delta_{-1} & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_{n-2} & \cdots & \Delta_{-1} \end{vmatrix}$$

This is just the NC Jacobi identity!

Some exact solutions

- We could generate various solutions of NC ASDYM eq. from a simple seed solution Δ_0 by using the previous Backlund trf.

A seed solution:

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}''$$

→NC instantons (special to NC spaces)

$$\Delta_0 = \exp(\,linear\ \ of\ \ z, \tilde{z}, w, \tilde{w})$$

→NC Non-Linear plane-waves
(new solutions beyond ADHM)
D-brane interpretation??

A compact formula of J-matrix

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

$$= \begin{vmatrix} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-n} & \Delta_{-n} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-n} & \Delta_{1-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \boxed{0} & \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{vmatrix}$$

- **J:gauge-invariant → The Backlund trf. is not just a gauge trf. but a non-trivial one!**

4. Conclusion and Discussion

NC integrable eqs (ASDYM) in higher-dim.

- ADHM (OK)
- Twistor (OK)
- Backlund trf (OK), Symmetry (Next)

Quasi-determinants are important !

Profound relation ?? (via Ward conjecture)

NC integrable eqs (KdV) in lower-dims.

- Hierarchy (OK)
- Infinite conserved quantities (OK)
- Exact N-soliton solutions (OK)
- Symmetry (``tau-fcn'', Sato's theory) (Next)
- Behavior of solitons… (Next)

Application to N=2 string, new formulation of integrable systems, and integrability of N=4SYM??

Quasi-determinants are important !