Duality in 2d (2,2) Supersymmetric Non-Abelian Gauge Theories

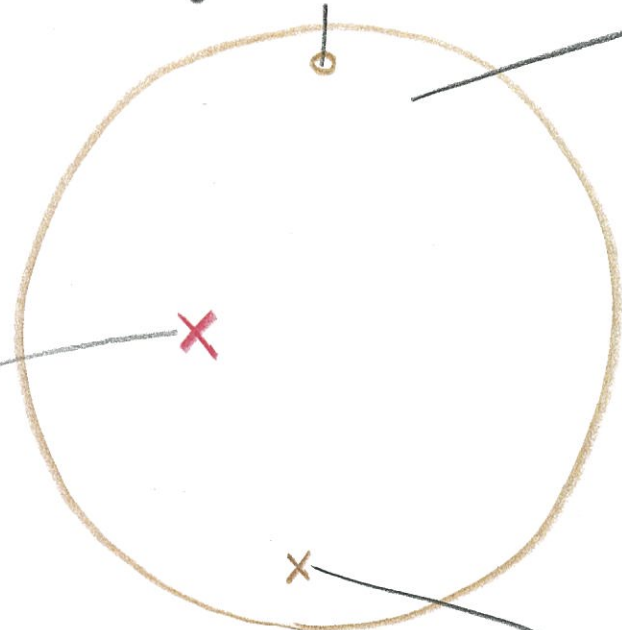
Kentaro Hori

Motivation

1991 Candelas, de la Ossa, Green, Parkes

$$\boxed{f(x_1, \cdots, x_5) \ \text{a Quintic polynomial}}$$

Large Volume limit



X_f : Calabi-Yau hypersurface

$$\{ f(x_1, \cdots, x_5) = 0 \} \subset \mathbb{CP}^4$$

Singular point

Landau-Ginzburg Orbifold (Gepner model)

$$W = f(x_1, \cdots, x_5)$$

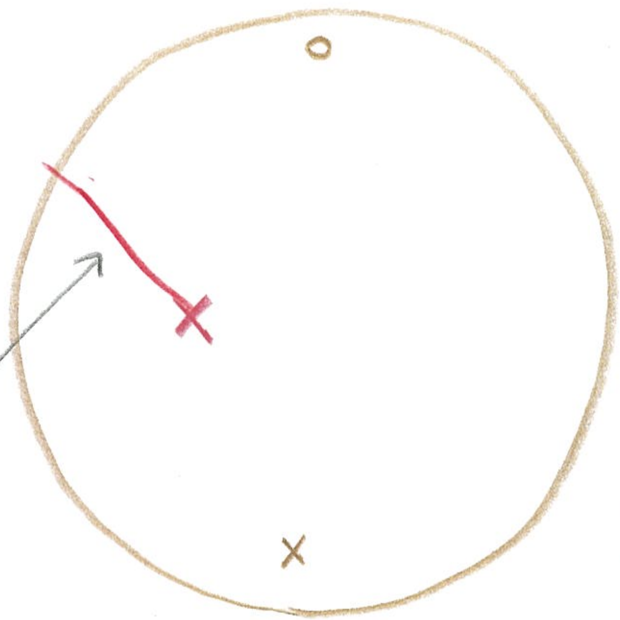
$$\text{mod } \mathbb{Z}_5 \ni \omega : (x_1, \cdots, x_5) \mapsto (\omega x_1, \cdots, \omega x_5)$$

1993 Witten "Linear σ-Model"

$G = U(1)$, $\quad \underset{-5}{P}, \underbrace{X_1, \dots, X_5}_{1}$, $\quad W = P f(X_1, \dots, X_5)$

D-term eqn:

$-5|p|^2 + |x|^2 = r$



$\underline{r>0}$ $\quad U(1) \xrightarrow{X \neq 0} \{1\}$

Classically $\rightsquigarrow$ CY X_f

$\underline{r<0}$ $\quad U(1) \xrightarrow{P \neq 0} \boxed{\mathbb{Z}_5}$

Classically $\rightsquigarrow$ LG Orbifold

$\exists$ Coulomb branch

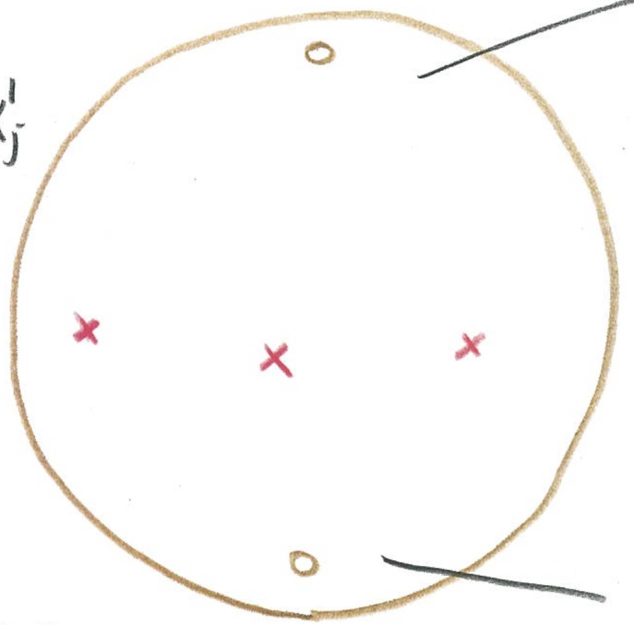
1998 Rødland

$$\boxed{A_k^{ij} = -A_k^{ji} \qquad i, j, k = 1, \cdots, 7}$$

$X_i = (X_i^a) \quad {}^{a=1,2}_{i=1,\cdots,7}$

$[X_i X_j] = X_i^1 X_j^2 - X_i^2 X_j^1$

Plücker coord



X_A C.I. CY^3 in $G(2,7)$

$$= \frac{\left\{ X \;\middle|\; \sum_{ij} A_k^{ij} [X_i X_j] = 0 \;\; \forall k \right\}}{GL(2,\mathbb{C})}$$

$p^i \quad i = 1, \cdots, 7$

$A(p) = (A^{ij}(p))$ 7×7 antisym

$$A^{ij}(p) = \sum_k A_k^{ij} p^k$$

Y_A Pfaffian CY^3 in $\mathbb{CP}^6$

$$= \{ p \mid \text{rank}\, A(p) \leq 4 \} / \mathbb{C}^\times$$

× : Singular points

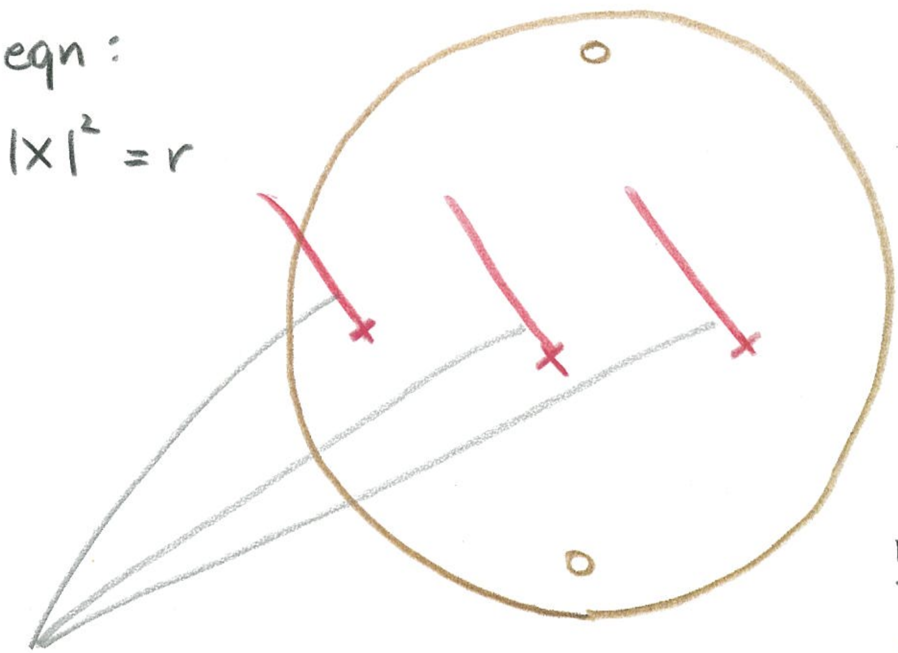
2006 KH - D. Tong

LσM

$$G = U(2), \quad \underbrace{P^1, \cdots, P^7}_{\det^{-1}}, \quad \underbrace{X_1, \cdots, X_7}_{\mathbb{2}}, \quad W = \sum_{ijk} A_k^{ij} P^k [X_i X_j]$$

D-term eqn:

$$-|p|^2 + |X|^2 = r$$



$\exists$ Coulomb branch

$\underline{r>0}$: $U(2) \xrightarrow{X \neq 0} \{1\}$

Classically $\rightsquigarrow$ X_A

$\underline{r<0}$: $U(2) \xrightarrow{P \neq 0} \boxed{SU(2)}$

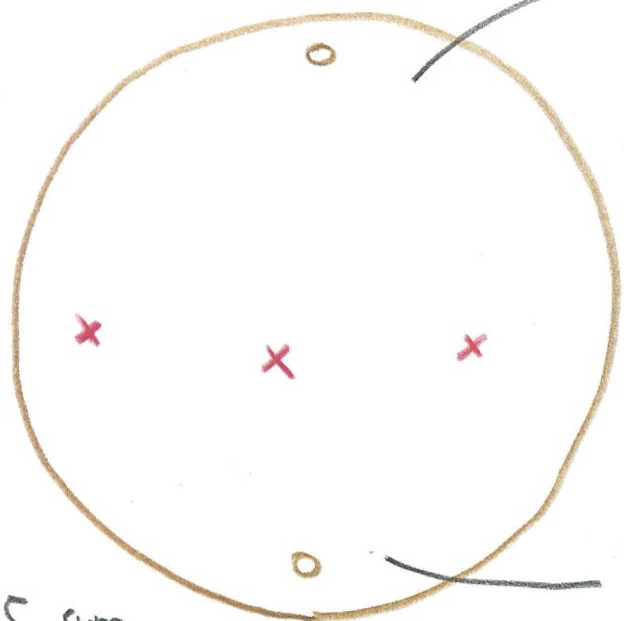
$\boxed{\text{Quantum}}$ $\rightsquigarrow$ Y_A

2011 Hosono-Takagi

$$\boxed{S^{ij}_k = S^{jk}_k \quad i,j,k=1,\dots,5}$$

$X_i = (X^a_i) \quad a=1,2 \quad i=1,\dots,5$

$(X_i X_j) = X^1_i X^1_j + X^2_i X^2_j$



X_S Reye Congruence

(a CICY^3 in $\mathbb{CP}^4 \times \mathbb{CP}^4 / \mathbb{Z}_2^{\text{exch.}}$)

$$= \frac{\{ x \mid \sum_{ij} S^{ij}_k (X_i X_j) = 0 \ \forall k \}}{(\mathbb{C}^\times \times O(2,\mathbb{C})) / \{(\pm 1, \pm \mathbb{1}_2)\}}$$

$p^{i=1,\dots,5}$

$S(p) = (S^{ij}(p)) \quad 5\times 5$ sym.

$S^{ij}(p) = \sum_k S^{ij}_k p^k$

$\times$: Singular points

$\widetilde{Y}_S$ ramified double cover

$$\downarrow \mathbb{Z}_2$$

$Y_S = \{ p \mid \operatorname{rank} S(p) \le 4 \} / \mathbb{C}^\times \subset \mathbb{CP}^4$

$\cup$

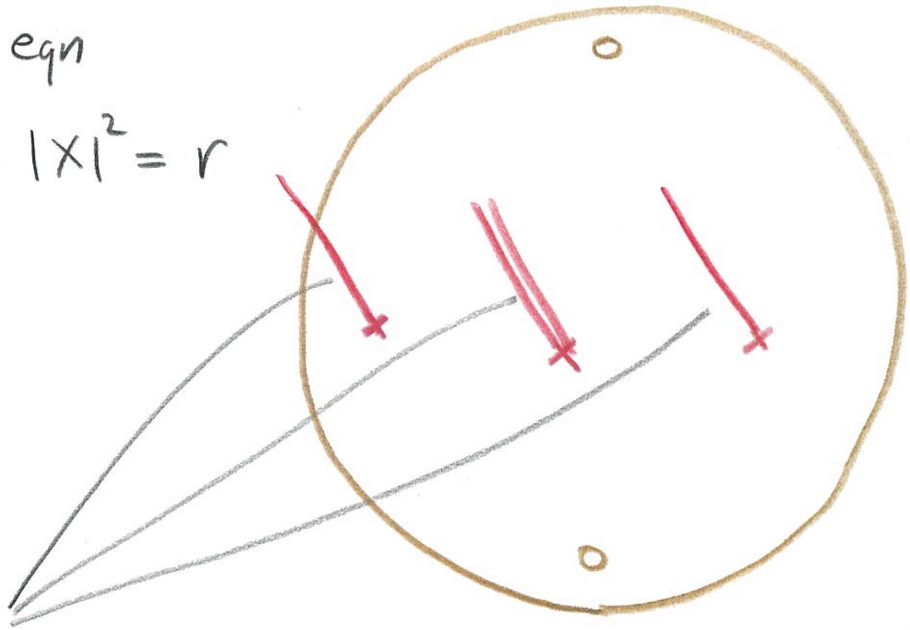
ram. along $C_S = \{ p \mid \operatorname{rank} S(p) \le 3 \} / \mathbb{C}^\times$

2011 KH L σ M

$G = \dfrac{U(1)\times O(2)}{\{(\pm 1, \pm \mathbb{1}_2)\}}$, $\underbrace{P^1,\dots,P^5}_{(-2,\mathbf{1})}$, $\underbrace{X_1,\dots,X_5}_{(1,\mathbf{2})}$, $W = \sum_{ijk} S^{ij}_{k} P^k (X_i X_j)$

D-term eqn

$-2|P|^2 + |X|^2 = r$



$\exists$ <u>Coulomb branch</u>

$\underline{r>0}$: $G \xrightarrow{X\neq 0} \{1\}$

Classically $\rightsquigarrow$ X_S

$\underline{r<0}$: $G \xrightarrow{P\neq 0} \boxed{O(2)}$

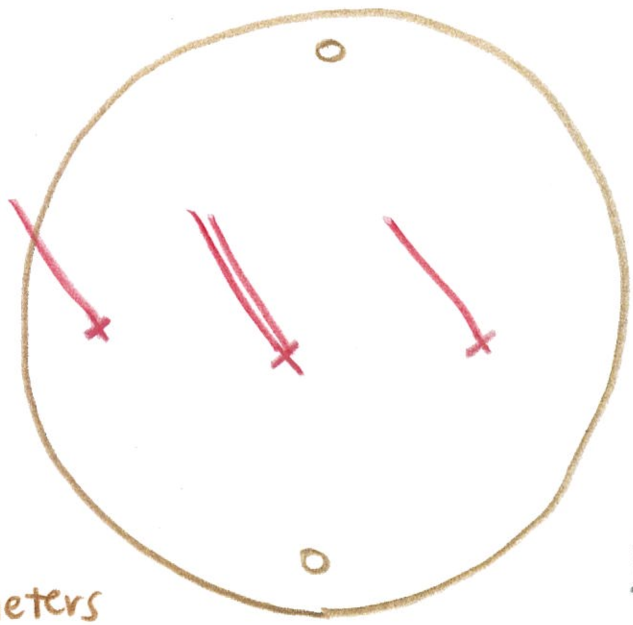
Quantum $\rightsquigarrow$ $\tilde{Y}_S$

$\exists$ Dual Linear σ-Model (for Hosono-Takagi)

$$\tilde{G} = \frac{U(1)\times SO(4)}{\{(\pm 1, \pm \mathbb{1}_4)\}}, \quad \underbrace{p^1,\dots,p^5}_{(-2,\mathbf{1})}, \underbrace{\tilde{x}^1,\dots,\tilde{x}^5}_{(-1,\mathbf{4})}, \underbrace{\overset{s_{ji}}{\overset{\shortparallel}{s_{ij}}}}_{(1,\mathbf{1})}, \qquad W = \sum s_{ij}(\tilde{x}^i \tilde{x}^j) + \sum S_k^{ij} p^k s_{ij}$$

D-term eqn

$$-2|p|^2 - |\tilde{x}|^2 + |s|^2 = \tilde{r}$$



$\underline{\tilde{r} > 0}$: $\tilde{G} \xrightarrow{s \neq 0} SO(4)$

$\xrightarrow[\rightsquigarrow]{\text{Quantum}} X_S$

$\underline{\tilde{r} < 0}$: $\tilde{G} \xrightarrow{\substack{\tilde{x} \neq 0 \\ p \neq 0}} \boxed{\{1\}}$

$\boxed{\text{Classically}} \rightsquigarrow \tilde{Y}_S$

Relation of parameters

$$\begin{cases} r = \tilde{r} + 5\log 2 \\ \theta \equiv \tilde{\theta} + \pi \end{cases}$$

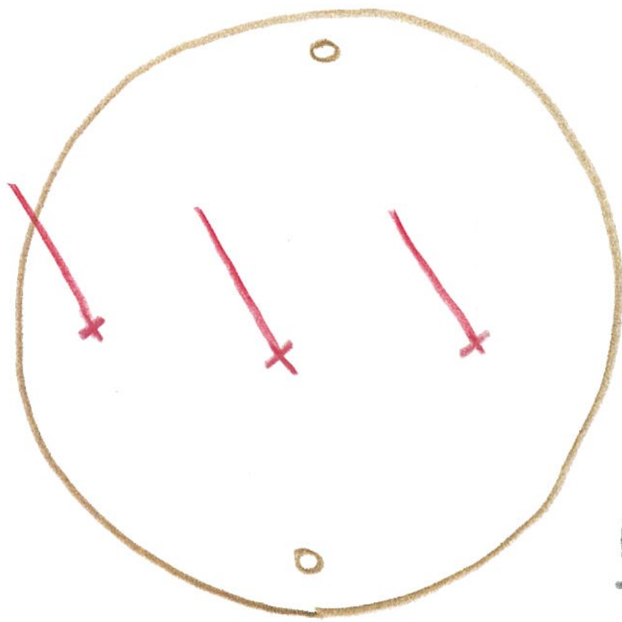
Dual Linear σ- Model (for Rødland)

$$\tilde{G} = \frac{U(1)\times USp(4)}{\{(\pm 1, \pm \mathbb{1}_4)\}}, \quad \underbrace{p^1,\dots,p^7}_{(-2,\,\mathbb{1})}, \quad \underbrace{\tilde{x}^1,\dots,\tilde{x}^7}_{(-1,\,\mathbb{4})}, \quad \underbrace{a_{ij}}_{(1,\,\mathbb{1})}, \quad a_{ij} = -a_{ji}$$

$$W = \sum a_{ij}[\tilde{x}^i \tilde{x}^j] + \sum A^{ij}_k\, p^k a_{ij}$$

D-term eqn

$$-2|p|^2 - |\tilde{x}|^2 + |a|^2 = \tilde{r}$$



$\underline{\tilde{r} > 0}$: $\tilde{G} \xrightarrow{a \neq 0} USp(4)$

$\xrightarrow{\text{Quantum}} X_A$

$\underline{\tilde{r} < 0}$: $\tilde{G} \xrightarrow{\substack{\tilde{x} \neq 0 \\ p \neq 0}} \boxed{\{1\}}$

$\xrightarrow{\boxed{\text{Classically}}} Y_A$

Relation

$$\begin{cases} r = \tilde{r} \\ \theta \equiv \tilde{\theta} \end{cases}$$

$\underline{\mathcal{M}_K \ni x, y} \Rightarrow D_B(x) \cong D_B(y)$

$\boxed{\text{Equivalence of Categories of B-branes}}$

- $D^b(X_f) \cong MF_{\mathbb{Z}_5}(f)$ Orlov 2005

- $D^b(X_A) \cong D^b(Y_A)$ Borisov-Caldararu, Kuznetsov 2006

- $D^b(X_S) \cong D^b(\widetilde{Y}_S)$ Hosono-Takagi 2011

<u>Note</u> $X_A \not\cong Y_A$, $X_S \not\cong \widetilde{Y}_S$ Birationally inequivalent

The Models

$k = 1, 2, 3, \dots,$ $\quad N = 0, 1, 2, \dots.$

$$\boxed{\begin{array}{l} \text{Gauge group } \; G = O(k) \text{ or } SO(k) \\ \text{Matters : } N \text{ fields in } \mathbb{C}^k \quad X_1, \dots, X_N \\ \text{Superpotential : } \; W = 0 \end{array}}$$

- $O(k) = SO(k) \rtimes \mathbb{Z}_2$ $\cdots$ $\mathbb{Z}_2$ orbifold of $SO(k)$ theory

  2 possibilities $\;O_+(k),\; O_-(k).$

- $k \geq 3$: $\pi_1(G) \cong \mathbb{Z}_2$ $\dashrightarrow$ mod 2 θ-angle, No F.I.

- $k = 2$: $SO(2)$: $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, F.I. $\in \mathbb{R}$; $\quad O_\pm(2)$: $\theta = 0$ or π, No F.I.

$k = 2, 4, 6, \cdots,$ $\quad N = 0, 1, 2, \cdots.$

$$\boxed{\begin{array}{l} \text{Gauge group } G = USp(k) \\ \text{Matters: } N \text{ fields in } \mathbb{C}^k \quad x_1, \cdots, x_N \\ \text{Superpotential : } W = 0 \end{array}}$$

$\pi_1(G) = \{0\} \dashrightarrow$ No θ, No F.I.

Regularity

To find the nature of low energy theory,

Let's first look at $\{U=0\}$.

· scalar fields : σ in vector, X in chiral

· Potential $U(\sigma, X) = \frac{1}{2g^2}\left|[\sigma,\sigma^\dagger]\right|^2 + \frac{1}{2}|\sigma X|^2 + \frac{1}{2}|\sigma^\dagger X|^2 + \frac{g^2}{2}|\mu(X)|^2$

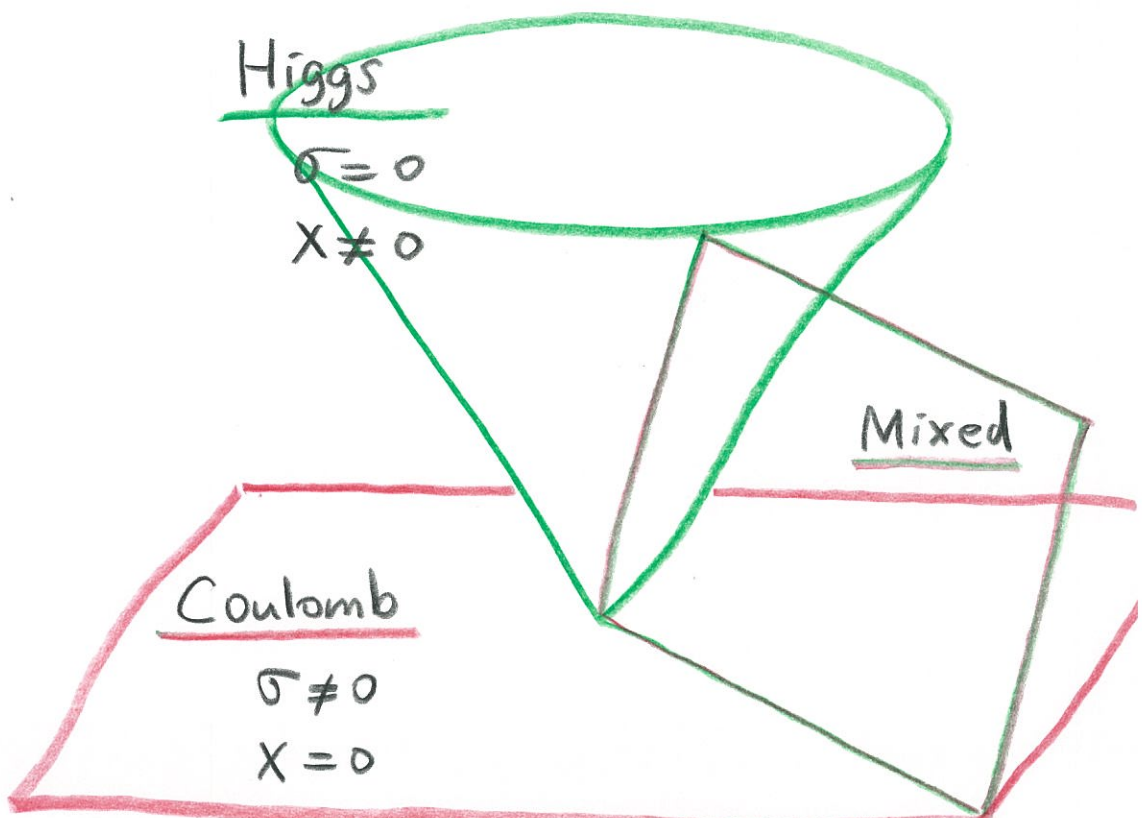
$U=0$: each term $=0$

$[\sigma,\sigma^\dagger]=0,\quad \sigma X = \sigma^\dagger X = 0,\quad \mu(X)=0$

$\uparrow$

e.g. $O(k)$ $k=2l$

$$\sigma = \begin{pmatrix} 0 & -\sigma_1 & & & \\ \sigma_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\sigma_l \\ & & & \sigma_l & 0 \end{pmatrix}$$



Problem : Non-compactness.

Higgs branch may be lifted by turning on $W(X)$,
but that does not change Coulomb branch.

In fact, $\exists$ quantum correction to U on the Coulomb.

$\sigma \neq 0$: $G \longrightarrow U(1)^{\ell}$ Copies of Maxwell theory

$$\boxed{\Delta U = \frac{g^2}{2} |\theta_{eff}|^2}$$

Electro-Static Energy

$G = O(k)$ or $SO(k)$: $\theta_{eff} \underset{2\pi\mathbb{Z}}{\equiv} \pi(N-k) + \theta_0$ $\left(\begin{array}{l} \theta_0 = 0 \text{ or } \pi : \\ \text{mod } 2\ \theta\text{-angle} \end{array} \right)$

$G = USp(k)$: $\theta_{eff} \underset{2\pi\mathbb{Z}}{\equiv} \pi(N-k)$

Thus, the Coulomb and the mixed branches are lifted by quantum correction if

$$O(k), SO(k): \quad N-k \text{ odd}, \quad \theta_0 \equiv 0$$

$$N-k \text{ even}, \quad \theta_0 \equiv \pi$$

$$USp(k) \quad : \quad N-k \text{ odd}.$$

Such theories are called regular.

Results

$G = O_\pm(k),\ SO(k)\ ,\ x_1, \cdots, x_N$ in $\mathbb{C}^k$, $W=0$

- $N \leq k-2$: ~~SUSY~~

↓ regular

- $N = k-1$: Free theory of mesons $(x_i x_j) = \sum_{a=1}^{k} x_i^a x_j^a$

  $\mathbb{C}^{\frac{N(N+1)}{2}}$ for $SO(k), O_-(k)$, $\mathbb{C}^{\frac{N(N+1)}{2}} \sqcup \mathbb{C}^{\frac{N(N+1)}{2}}$ for $O_+(k)$.

- $N \geq k$: $\exists$ duality

$$\boxed{\begin{aligned} G = O_+(k) &\longleftrightarrow \tilde{G} = SO(N-k+1) \\ SO(k) &\longleftrightarrow O_+(N-k+1) \\ O_-(k) &\longleftrightarrow O_-(N-k+1) \end{aligned}}$$

The dual theory has N fields $\tilde{x}^1, \cdots, \tilde{x}^N$ in $\mathbb{C}^{N-k+1}$

$\frac{N(N+1)}{2}$ singlets $S_{ij} = S_{ji}$,

$$W = \sum_{i,j} S_{ij} (\tilde{x}^i \tilde{x}^j)$$

- mesons $(X_i X_j)$ $\longleftrightarrow$ singlets S_{ij}

- $G = SO(k)$ vs $\tilde{G} = O_+(N-k+1)$

$\mathbb{Z}_2$ global symmetry $O(k)/SO(k)$ $\longleftrightarrow$ Quantum $\mathbb{Z}_2$ symmetry of the $\mathbb{Z}_2$ orbifold

baryons $[X_{i_1} \cdots X_{i_k}]$ $\longleftrightarrow$ twist operators

- $G = O_+(k)$ vs $\tilde{G} = SO(N-k+1)$

Similar

- $G = O_-(k)$ vs $\tilde{G} = O_-(N-k+1)$

quantum $\mathbb{Z}_2$ symmetry $\longleftrightarrow$ quantum $\mathbb{Z}_2 (-1)^F$ symmetry.

$G = USp(k)$, $X_1, \cdots, X_N$ in $\mathbb{C}^k$, $W=0$

- $N \leq k$: ~~SUSY~~

$\downarrow$ regular

- $N = k+1$: Free theory of mesons $[X_i X_j] = \sum_{a,b} X_i^a J_{ab} X_j^b$

  i.e. CFT of $\mathbb{C}^{\frac{N(N-1)}{2}}$.

- $N \geq k+3$: $\exists$ dual theory

$$\boxed{\begin{array}{l} \tilde{G} = USp(N-k-1) \\ N \text{ fields } \tilde{X}^1, \cdots, \tilde{X}^N \text{ in } \mathbb{C}^{N-k-1} \\ \frac{N(N-1)}{2} \text{ singlets } \quad a_{ij} = -a_{ji} \\ W = \sum_{i,j} a_{ij} [\tilde{X}^i \tilde{X}^j] \end{array}}$$

mesons $[X_i X_j] \longleftrightarrow$ singlets a_{ij}

Tests of Duality

- Check at low values of k, N $\left(\text{cf. } O(1) = \mathbb{Z}_2 ,\ SO(1) = \{1\} \right)$

- Symmetry

O, SO

	$SU(N)$	$U(1)_B$	$U(1)_V$	$U(1)_A$
X	$\square$	1	0	0
$\tilde{X}$	$\overline{\square}$	-1	1	0
S	$\square\!\square$	2	0	0

match!

$$\hat{c} = kN - \frac{k(k-1)}{2}$$

$$\hat{c} = \frac{N(N+1)}{2} - \frac{(N-k+1)(N-k)}{2}$$

USp

	$SU(N)$	$U(1)_B$	$U(1)_V$	$U(1)_A$
X	$\square$	1	0	0
$\tilde{X}$	$\overline{\square}$	-1	1	0
a	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	2	0	0

match!

$$\hat{c} = kN - \frac{k(k+1)}{2}$$

$$\hat{c} = \frac{N(N-1)}{2} - \frac{(N-k-1)(N-k)}{2}$$

- Born-Oppenheimer for dual.

$$\left.\begin{aligned} &\mathrm{corank}(S) \leq (N-k+1)-2 : \cancel{\text{SUSY}} \\ &\mathrm{corank}(S) = (N-k+1)-1 : \text{Free theory of mesons} \end{aligned}\right\} \Rightarrow \underline{\mathrm{rank}(S) \leq k}$$

$$\left.\begin{aligned} &\mathrm{corank}(a) \leq (N-k-1) : \cancel{\text{SUSY}} \\ &\mathrm{corank}(a) = (N-k-1)+1 : \text{Free theory of mesons} \end{aligned}\right\} \Rightarrow \underline{\mathrm{rank}(a) \leq k}$$

- Flow in complex mass

$$\begin{array}{ccc} O_+(k),\ N & \longleftrightarrow & SO(N-k+1),\ N \\ \Delta W = m(X_N X_N)\big\downarrow & & \big\downarrow \Delta W = m S_{NN} \rightsquigarrow (\tilde{X}^N \tilde{X}^N) + m = 0 \\ O_+(k),\ N-1 & & SO(N-k),\ N-1 \end{array}$$

- Vacuum counting with twisted masses. ✓

- Comparison of chiral rings : (c,c) & (a,c). ✓