# *W*-curves and Mirror Symmetry

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# An Orbifolded LG A-model

The theory of Fan-Jarvis-Ruan-Witten

- provides an orbifolded LG A-Model for each quasi-homogeneous singularity $W$
- is known to satisfy a form of LG-LG mirror symmetry
- fits into the LG-CY correspondence

# Conditions on $W$

By *singularity* we mean

- $W \in \mathbb{C}[x_1, \ldots, x_N]$ is quasi-homogeneous with weights $q_1, \ldots, q_N \in \mathbb{Q} \cap [0, 1/2]$
- $W$ is non-degenerate (isolated singularity at origin)
- The weights of $W$ are unique (i.e., no term of the form $xy$)

# Examples of Singularities

$$
\begin{array}{lcll}
A_n & = & x^{n+1} & \text{weight } q_x = \frac{1}{(n+1)} \\
E_6 & = & x^3 + y^4 & \text{weights } q_x = 1/3 \qquad q_y = 1/4 \\
E_7 & = & x^3 + xy^3 & \text{weights } q_x = 1/3 \qquad q_y = 2/9 \\
E_8 & = & x^3 + y^5 & \text{weights } q_x = 1/3 \qquad q_y = 1/5 \\
D_n & = & x^{n-1} + xy^2 & \text{weights } q_x = \frac{1}{(n-1)} \qquad q_y = \frac{(n-1)}{2n} \\
P_8 & = & x^3 + y^3 + z^3 & \text{weights } q_x = q_y = q_z = 1/3
\end{array}
$$

# Automorphism groups

There is a canonical automorphism of $W$ defined by

$$J = (\exp(2\pi i q_1, \ldots, 2\pi i q_N) \in (\mathbb{C}^*)^N.$$

The largest group of automorphisms we allow is

$$G_{max} = \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N | W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$$

A group $G$ is *admissible* if

$$J \in G \leq G_{max}$$

# An orbifold Landau-Ginzburg A-model

### Theorem (Fan, Jarvis, Ruan (after Witten))

*For each nondegenerate W and admissible G there is a*

- *Moduli space* $\mathscr{W}^{W,G}_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}$
- *State space* $\mathscr{H}_{W,G} = \bigoplus_{\gamma \in G} \mathscr{H}_\gamma$
- *Pairing* $\langle\ ,\ \rangle : \mathscr{H}_\gamma \times \mathscr{H}_{\gamma^{-1}} \longrightarrow \mathbb{C}$
- *Virtual Cycle* $\left[\mathscr{W}^{W,G}_{g,k}\right]^{vir}$
- *Cohomological Field Theory* $\{\Lambda^{W,G}_{g,k}\},\ \mathscr{H},\ \langle\ ,\ \rangle$

# Consequences of a CohFT

Associated to the CohFT we get

- $g = 0, n = 3$ Frobenius algebra $\mathscr{H}_{W,G}, \star$
- $g = 0, n \geq 3$ Frobenius manifold
- $g \geq 0$ potential $\Phi^{FJRW}_{W,G}$

# LG B-model

Associated to each singularity $W$ we have

- Frobenius algebra: Milnor ring $\mathbb{C}[x_1, \ldots, x_N]/(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N})$
- Saito Frobenius manifold associated to $W$ (semi-simple)
- Givental formal potential $\Phi_W^{Givental}$
- In some cases:
  Kac-Wakimoto/Drinfeld-Sokolov integrable hierarchy

# LG-LG mirror symmetry

| A-model (FJRW) | | B-model | | | |
|---|---|---|---|---|---|
| | | $g=0$ | $g=0$ | | |
| | | $n=3$ | $n\geq 3$ | $g\geq 0$ | |
| Singu-larity | Group | Milnor Ring | Saito Frob Manifold | Givental Potential | DS/KW int hier |
| $A_n$ | $\langle J\rangle = G_{max}$ | $A_n$ | $A_n$ [JKV] | $A_n$ [L,FSZ] | $A_n$ |
| $E_6$ | $\langle J\rangle = G_{max}$ | $E_6$ | $E_6$ | $E_6$ | $E_6$ |
| $E_7$ | $\langle J\rangle = G_{max}$ | $E_7$ | $E_7$ | $E_7$ | $E_7$ |
| $E_8$ | $\langle J\rangle = G_{max}$ | $E_8$ | $E_8$ | $E_8$ | $E_8$ |
| $D_n$ $n$ even | $\langle J\rangle$ $[G_{max}:\langle J\rangle]=2$ | $D_n$ $[D_n^T/\mathbb{Z}_2]$ | $D_n$ $[D_n^T/\mathbb{Z}_2]$ | $D_n$ ? | $D_n$ ? |
| $D_n$ $n$ even | $G_{max}$ | $D_n^T$ | $D_n^T$ | $D_n^T$ | $D_n^T$ |
| $D_n$ $n$ odd | $G_{max}$ | $D_n^T$ | $D_n^T$ | $D_n^T$ | $D_n^T$ |
| $D_n^T$ | $\langle J\rangle = G_{max}$ | $D_n$ | $D_n$ | $D_n$ | $D_n$ |

# Berlund-Hübsch duality

For each singularity

$$W = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_i^{a_{ij}}$$

with the same number of monomials as variables (invertible), just transpose the exponent matrix:

$$A := (a_{ij}) \Rightarrow A^T = (a_{ji})$$

to get

$$W^T = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_i^{a_{ji}}$$

- $A_n, E_{6,7,8}, P_8$ are all self-dual
- $D_n^T = x^{n-1}y + y^2 \cong A_{2n-3}$

# LG-LG Conjecture/Expectation

Krawitz defines $G^T$ in such a way that

- $|G^T| = [G_{max} : G]$
- $G^T \leq SL_N$ iff $J \in G$

Orbifold mirror:

- For Frobenius algebras:
  FJRW (A-model) for $W/G$ is isomorphic to IV-K-K orbifold Milnor ring for $W^T/G^T$
- Expect same for Frobenius manifolds:
  (Orbifolded Frobenius manifold is not yet fully defined)

# LG-LG mirror symmetry

| A-model (FJRW) | | B-model | | |
|---|---|---|---|---|
| $W$ | $G$ | $g=0$<br>$n=3$<br>Milnor Ring | $g=0$<br>$n\geq 3$<br>Frob Manifold | $g\geq 0$<br>Givental Potential |
| $P_8$ | $G_{max}$ | $P_8$ | $P_8$ [KS] | $P_8$ [KS] |
| $X_9$ | $G_{max}$ | $X_9$ | $X_9$ [KS] | $X_9$ [KS] |
| $J_{10}$ | $G_{max}$ | $J_{10}$ | $J_{10}$ [KS] | $J_{10}$ [KS] |
| $W$ | $G_{max}$ | $W^T$ [AKS] | $W^T$? | $W^T$? |
| $W$<br>for most<br>inv $W$ | $G < G_{max}$ | $\left[W^T/G^T\right]$ [FJJ] | $\left[W^T/G^T\right]$? | $\left[W^T/G^T\right]$?<br>(if SS) |