$(0, 2)$ Quantum Cohomology

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Outline

1. Background
   - Quantum Cohomology
   - The half-twisted model
   - The Gauged Linear Sigma Model and Toric Geometry

2. Correlation Functions and Quantum Cohomology
   - Correlation Functions
   - Quantum Cohomology

Abstract

In this talk, a mathematical definition is given of the topological correlation functions of a $(0, 2)$ gauged linear sigma model in the geometric phase of a neighborhood of the $(2, 2)$ locus in moduli. The geometric data determining the model is a smooth toric variety X and a deformation E of its tangent bundle TX. This definition is consistent with the known results of physics and leads to a proof of the existence of a quantum cohomology ring in complete generality, extending known results of physics.

The A model

- Consider the topological A-model on a compact Kähler manifold (X, g) (no coupling to gravity, a TQFT)
- Observables $H^*(X)$
- Correlation functions $\langle \omega_1, \ldots, \omega_n \rangle_\beta,\ \omega_i \in H^{2k_i}(X),\ \beta \in H_2(X, \mathbf{Z})$
- Virtual dimension $D = c_1(X) \cdot \beta + \dim X, \qquad \sum k_i = D$
- $\langle \omega_1, \ldots, \omega_n \rangle = \sum_\beta \langle \omega_1, \ldots, \omega_n \rangle_\beta q^\beta,\ q^\beta = \exp(\int_\beta (B + ig))$, where B is the B-field

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A Model Lagrangian

$$\mathcal{L} = \frac{1}{\alpha'}\int_\Sigma d^2z\,\Big((g_{\mu\nu}+iB_{\mu\nu})\,\partial\phi^\mu\overline{\partial}\phi^\nu \;+\; \frac{i}{2}g_{\mu\nu}\psi_+^\mu D_{\overline{z}}\psi_+^\nu + \frac{i}{2}g_{\mu\nu}\psi_-^\mu D_z\psi_-^\nu \;+\; R_{i\bar{\jmath}k\bar{l}}\psi_+^i\psi_+^{\bar{\jmath}}\psi_-^k\psi_-^{\bar{l}}\Big)$$

The fields have been twisted from the usual sigma model so that

$$\psi_+^i \in C^\infty(\phi^* TX), \qquad \psi_-^i \in C^\infty\left(\overline{K}_\Sigma \otimes \left(\phi^*\overline{TX}\right)^*\right),$$

$$\psi_+^{\bar{\imath}} \in C^\infty\left(K_\Sigma \otimes \left(\phi^*\overline{TX}\right)^*\right), \qquad \psi_-^{\bar{\imath}} \in C^\infty\left(\phi^*\overline{TX}\right).$$

where $\phi:\Sigma\to X$ is the map from the worldsheet to X.

Quantum Cohomology

- The observables are in 1-1 correspondence with elements of $H^*(X)$, generating the *quantum cohomology ring* by operator product
- In QFT, an operator is zero by definition if after inserting it into a correlation function, all correlation functions vanish after arbitrary additional insertions
- The quantum cohomology ring is the algebra generated by a basis for the cohomology, modulo those products which are zero as operators in the above sense

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Quantum Cohomology Relations

In particular, a quantum cohomology relation of the form

$$\prod_{i=1}^{r} \alpha_i = q^\gamma \prod_{j=1}^{s} \eta_j, \;\; \alpha_i, \eta_j \in H^*(X)$$

is equivalent to the identities

$$\langle \alpha_1, \ldots, \alpha_r, \omega_1, \ldots, \omega_n \rangle = q^\gamma \langle \eta_1, \ldots, \eta_s, \omega_1, \ldots, \omega_n \rangle$$

for any $\omega_1, \ldots, \omega_n$, which is in turn equivalent to

$$\langle \alpha_1, \ldots, \alpha_r, \omega_1, \ldots, \omega_n \rangle_{\beta+\gamma} = \langle \eta_1, \ldots, \eta_s, \omega_1, \ldots, \omega_n \rangle_\beta$$

for any ω_I and β.

The A/2 model

The model is a “half-twist" of the $(0,2)$ nonlinear sigma model described by the Lagrangian

$$\mathcal{L} = \frac{1}{\alpha'}\int_\Sigma d^2z\,\Big((g_{\mu\nu}+iB_{\mu\nu})\,\partial\phi^\mu\overline{\partial}\phi^\nu \;+\; \frac{i}{2}g_{\mu\nu}\psi_+^\mu D_{\overline{z}}\psi_+^\nu \;+\; \frac{i}{2}h_{\alpha\beta}\lambda_-^\alpha D_z\lambda_-^\beta \;+\; F_{i\overline{\jmath}a\overline{b}}\psi_+^i\psi_+^{\overline{\jmath}}\lambda_-^a\lambda_-^{\overline{b}}\Big)$$

with field content

$$\psi_+^i \in C^\infty(\phi^*TX), \qquad \lambda_-^a \in C^\infty\left(\overline{K_\Sigma}\otimes\phi^*\overline{E}^*\right),$$
$$\psi_+^{\overline{\imath}} \in C^\infty\left(K_\Sigma\otimes(\phi^*TX)^*\right), \qquad \lambda_-^{\overline{a}} \in C^\infty\left(\phi^*\overline{E}\right),$$

where E is a holomorphic vector bundle on X.

Anomaly Cancellation

- Anomaly cancellation requires $\Lambda^{\text{top}} E^* \cong K_X, \ \text{ch}_2(E) = \text{ch}_2(TX)$
- We call such a bundle *E omalous*
- Deformations *E* of *TX* are omalous

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Observables and Correlation Functions

- The half-twisted model is not topological, but has a sector in which the OPE closes [Adams-Distler-Ernebjerg].
- In this sector, the observables are in 1-1 correspondence with the cohomology ring $H^*(X, \Lambda^* E^*)$, which we call the *polymology* of (X, E).
- Since $\Lambda^{\mathrm{top}} E^* \simeq K_X$, we have $H^{\mathrm{top}}(X, \Lambda^{\mathrm{top}} E^*) \simeq \mathbf{C}$, providing a mathematical definition of classical correlation functions.
- Quantum corrections deform the classical polymology ring. In many situations, we know from physics that this deformation is an associative ring, the $(0, 2)$ quantum cohomology ring.

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Calling all Mathematicians

- A rigorous mathematical definition of correlation functions in the A/2 theory could produce a powerful method for computing Yukawa couplings in heterotic string theory
- Alas, this is beyond today's technology. But we can rigorously define and easily compute correlation functions exactly in the analogous gauged linear sigma model, when E is a deformation of TX.
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The Gauged Linear Sigma Model

- The $(0, 2)$ gauged linear sigma model (GLSM) is a 2D QFT with $(0, 2)$ SUSY. It is obtained from the $(2, 2)$ GLSM by decomposing the $(2, 2)$ multiplets into $(0, 2)$ multiplets, then varying the $(0, 2)$ multiplets independently.
- In a certain regime of FI parameters, the moduli space of vacua corresponds to a toric variety X and holomorphic vector bundle E on X.
- To be expeditious, I start with a smooth projective toric variety X and a deformation E of the tangent bundle TX, then engineer a $(0, 2)$ GLSM from that data.

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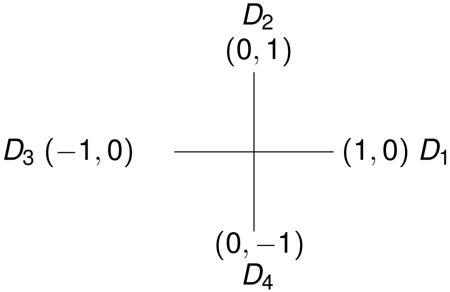
Toric Geometry: Notation

- T: a torus $T \simeq (\mathbf{C}^*)^r$
- $N = \mathrm{Hom}(\mathbf{C}^*, T) \simeq \mathbf{Z}^r$ the lattice of 1-parameter subgroups
- $M = \mathrm{Hom}(T, \mathbf{C}^*) \simeq \mathbf{Z}^r$ the lattice of characters.
- $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$: $(m \circ n)(t) = t^{\langle m,n \rangle}$
- Σ complete simplicial fan in $N_{\mathbf{R}} = N \otimes \mathbf{R}$.
- $X = X_\Sigma$ the associated complete toric variety, assumed smooth.
- $\Sigma(1)$: the set of 1-dimensional cones in Σ.
- $S = \mathbf{C}[x_\rho \mid \rho \in \Sigma(1)]$ homogeneous coordinate ring
- T-invariant divisor D_ρ defined by $x_\rho = 0$; $x_\rho \in H^0(\mathcal{O}(D_\rho))$
- $W = H^2(X) = \mathbf{C}^{\Sigma(1)}/(M \otimes \mathbf{C})$; divisors mod linear equivalence

Example: $\mathbf{P}^1 \times \mathbf{P}^1$



$$S = \mathbf{C}[x_1, x_2, x_3, x_4], \qquad ((x_1 : x_3), (x_2, x_4)) \in \mathbf{P}^1 \times \mathbf{P}^1$$

$$D_1 \sim D_3,\ D_2 \sim D_4,\ H_1 = [D_1] = [D_3],\ H_2 = [D_2] = [D_4]$$

$$W = H^2(X, \mathbf{C}) = \mathrm{span}(H_1, H_2)$$

Relation to GLSM

- There is an open set $U_\Sigma \subset \mathbf{C}^{\Sigma(1)}$ invariant under the complexification $G_\mathbf{C}$ of the gauge group G, so that $X_\Sigma = U_\Sigma / G_\mathbf{C}$
- The theory contains gauge fields for G
- Have a charged $(0,2)$ chiral field Φ_ρ for each $\rho \in \Sigma(1)$
- In the situation of a deformation of TX, also have fermi fields Λ_ρ with the same charges as those of Φ_ρ
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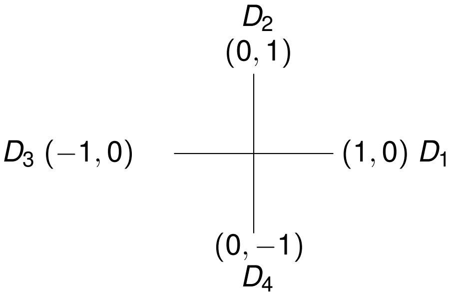

$$G = U(1) \times U(1)$$

	Φ_1	Φ_2	Φ_3	Φ_4
$U(1)_1$	1	0	1	0
$U(1)_2$	0	1	0	1

Tangent Bundle and its Deformations

In general,

$$0 \longrightarrow T^*X \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}(-D_\rho) \longrightarrow W\otimes\mathcal{O} \longrightarrow 0,$$

The rightmost nontrivial map is induced by the canonical sections $x_\rho \otimes [D_\rho]$ of $\mathcal{O}(D_\rho \otimes W)$.
Deformation of TX

$$0 \longrightarrow E^* \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}(-D_\rho) \longrightarrow W\otimes\mathcal{O} \longrightarrow 0,$$

by deforming $x_\rho \otimes [D_\rho]$ to sections $s_\rho \in \mathcal{O}(D_\rho \otimes W)$ sufficiently generic so that the kernel E^* is still a vector bundle.

Gauge Sectors

- Fix genus 0 worldsheet $\Sigma = \mathbf{P}^1$
- The theory forms sectors according to the topological type of the gauge bundle
- For $\mathbf{P}^1 \times \mathbf{P}^1$, assign chern classes (d, e) to $U(1) \times U(1)$ gauge bundle
- $\Phi_1, \Phi_3, \Lambda_1, \Lambda_3$ become global sections of $\mathcal{O}_{\mathbf{P}^1}(d)$, charge $(1, 0)$
- $\Phi_2, \Phi_4, \Lambda_2, \Lambda_4$ become global sections of $\mathcal{O}_{\mathbf{P}^1}(e)$, charge $(0, 1)$
- The zero modes of the Φ_i fill out the GLSM moduli space $X_{(d,e)} = \mathbf{P}^{2d+1} \times \mathbf{P}^{2e+1}$, exactly as in the (2, 2) situation
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- The zero modes of the Ψ_i fill out a deformation of the tangent bundle of $X_{(d,e)}$

The General Case

- In general, the topological types of the gauge bundle correspond to homology classes $\beta \in H_2(X, \mathbf{Z})$
- Have a GLSM moduli space X_β parametrized by the zero modes of the Φ_β
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Classical Correlation Functions

$$0 \longrightarrow E^* \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}(-D_\rho) \longrightarrow W \otimes \mathcal{O} \longrightarrow 0,$$

$H^0(\mathcal{O}(-D_\rho)) = H^1(\mathcal{O}(-D_\rho)) = 0$ gives

$$\psi : W = H^0(W \otimes \mathcal{O}) \simeq H^1(E^*),$$

hence cup product

$$\psi : \mathrm{Sym}^k W \to H^k(\Lambda^k E^*).$$

Fixing a normalization $\int_X : H^{top}(X, \Lambda^{top} E^*) \simeq \mathbf{C}$, for $P \in \mathrm{Sym}^{\dim(X)} W$, define the classical correlation function as

$$\langle P \rangle_0 = \int_X \psi(P)$$

Classical Correlation Functions, Continued

As we will see, the cup product can be computed explicitly by algebraic geometry. The method builds on the earlier computational methods of [K-Sharpe] and [Guffin-K] developed to verify a conjecture of [Adams-Basu-Sethi], while providing new viewpoints that elucidate more of the structure.

Classical Correlation Functions: $\mathbf{P}^1 \times \mathbf{P}^1$

- Put $Z = \oplus\mathcal{O}(-D_i) \simeq \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2$. Then the cup product is identified with the extension class of the generalized Koszul complex on the s_ρ

$$0 \to \Lambda^2 E^* \to \Lambda^2 Z \to Z \otimes W \to \mathrm{Sym}^2 W \otimes \mathcal{O} \to 0.$$

- This can be broken up into short exact sequences

$$0 \to \Lambda^2 E^* \to \Lambda^2 Z \to S_1 \to 0,$$

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- The cup product factors as $\mathrm{Sym}^2 W \to H^1(S_1) \to H^2(\Lambda^2 E^*)$
- We also see from $H^i(Z \otimes W) = 0$ that $\mathrm{Sym}^2 W \to H^1(S_1)$ is an isomorphism, while $H^1(S_1) \to H^2(\Lambda^2 E^*)$ is surjective and has kernel generated by $H^1(\Lambda^2 Z)$.

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$\mathbf{P}^1 \times \mathbf{P}^1$, Continued

- $Z = \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2$ gives $\Lambda^2 Z = \mathcal{O}(-2,0) \oplus \mathcal{O}(-1,-1)^4 \oplus \mathcal{O}(0,-2)$
- Only nonzero contributions $H^1(\mathcal{O}(-2,0))$ and $H^1(\mathcal{O}(0,-2))$ to $H^1(\Lambda^2 Z)$ arise from the respective pairs of divisors $\{D_1, D_3\}$; $\{D_2, D_4\}$, which do not intersect in X. More generally, these are the *primitive collections* of toric geometry.
- Chasing through the diagrams gives an explicit polynomial $Q \in \mathrm{Sym}^2 W$ associated with the generator of $H^1(\mathcal{O}(-2,0))$ and another $\tilde{Q} \in \mathrm{Sym}^2 W$ associated with the generator of $H^1(\mathcal{O}(0,-2))$, which lie in the kernel of the cup product.
- In summary, we have computed the polymology of (X, E) as

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Explicit Calculation

- Since $H^0(\mathcal{O}(D_1)) = H^0(\mathcal{O}(D_3))$ is spanned by x_1 and x_3, can write

$$s_1 = w_{11}x_1 + w_{13}x_3, \qquad s_3 = w_{31}x_1 + w_{33}x_3$$

for certain $w_{ij} \in W$.

- Putting

$$A = \begin{pmatrix} w_{11} & w_{13} \\ w_{31} & w_{33} \end{pmatrix}$$

we compute $Q = \det(A)$.

- The computation of $\tilde{Q}$ is analogous.
- Since the polymology is finite dimensional, $Q, \tilde{Q}$ have no common factors and it follows that the top part $\dim \mathrm{Sym}^2 W/(Q, \tilde{Q})$ of the polymology is one dimensional

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Conclusion of the Calculation

The computation of the classical correlation functions $\langle P \rangle_0$ with $P \in \mathrm{Sym}^2 W$ becomes trivial: take the image of P in the quotient $\mathrm{Sym}^2 W/(Q, \tilde{Q})$ and this is the unnormalized correlation function. Choose your favorite isomorphism of this 1 dimensional vector space with **C** if you want to normalize.

Classical Polymology, General Case

- In general, the same construction gives the classical polymology $H^*(X, \Lambda^* E^*)$ as a quotient $\mathrm{Sym}\, W/I$.
- To each primitive collection $\{D_{i_1}, \ldots, D_{i_k}\}$ which we index by $K = \{i_1, \ldots, i_k\}$, get a nonvanishing cohomology $H^{k-1}(X, \mathcal{O}(-\sum D_{i_j})) \simeq \mathbf{C}$ and a generator Q_K of I.
- Each Q_K is explicitly computable as a product of determinants of the linear coefficients of the s_ρ used to define E.
- I is simply the ideal generated by all of the Q_K, so is independent of nonlinear deformations!
- If $E = TX$, then Q_K is just the product $[D_{i_1}] \cdots [D_{i_k}]$ occuring in the definition of the Stanley-Reisner ideal. Thus our computation of the polymology reduces to the familiar toric description of $H^*(X)$, as it should.

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More Details of the Computation

- Primitive collections are compatible with linear equivalence: if K is a primitive collection and $\rho \in K$, then if $D_\rho \sim D_{\rho'}$ it can be shown that $\rho' \in K$. Each K can therefore be partitioned into a set T_K^+ of linear equivalence classes.
- Among the sections of $H^0(\mathcal{O}(D_\rho))$ are the sections expressed as a linear combination of the $x_{\rho'}$ for $D_{\rho'}$ in the linear equivalence class of D_ρ. The deformations of TX with these terms and no others have been called *linear deformations* in the physics literature. Fixing a linear equivalence class c, the linear terms associated with the s_ρ for all $\rho \in c$ form a square matrix. Let Q_c be its determinant.
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Nonclassical Correlation Functions

- Since X_β is a toric variety and E_β is a deformation of TX_β, we only have to describe its primitive collections K_β and the Q_{K_β} in terms of the classical data.
- Recall from [Morrison-Plesser] that the fan for X_β is obtained from that of X by replacing each edge $\rho \in \Sigma(1)$ with $h^0(D_\rho \cdot \beta) := h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D_\rho \cdot \beta))$ edges $\rho_1, \ldots, \rho_{h^0(D_\rho \cdot \beta)}$, keeping the same gauge group and charges, and charactering the fan by requiring that the primitive collections K_β of X_β are in 1-1 correspondence with the primitive collections K for X, associating to K the collection of divisors

$$K_\beta = \{\rho_i \mid \rho \in K\}.$$

- The result is $Q_{K_\beta} = \prod_{c \in T_K^+} Q_c^{h^0(D_c \cdot \beta)}$.

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Correlation Functions in sector β, naive attempt

Putting everything together, we learn that the polymology of X_β is $\mathrm{Sym}\, W/(I_\beta)$,

$$I_\beta = \left(\prod_{c \in T_K^+} Q_c^{h^0(D_c \cdot \beta)} \mid K \text{ a primitive collection for X} \right)$$

Example: $\mathbf{P}^1 \times \mathbf{P}^1$

- For $\beta = (d, e)$, have $h^0(H_1 \cdot \beta) = d + 1,\ h^0(H_2 \cdot \beta) = e + 1$, so the polynomial in sector β is

$$\mathrm{Sym}\, W / \left(Q^{d+1}, \tilde{Q}^{e+1} \right).$$

- This is all that is need to compute correlation functions as elements of the 1 dimensional vector space $\mathrm{Sym}^{2d+2e+2} W/(Q^{d+1}, \tilde{Q}^{e+1})$.

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A Problem

This is still not enough to compute correlation functions: $\dim X_\beta \neq c_1(X) \cdot \beta + \dim X$ in general. We need *four-fermi terms*, an analogue of the virtual fundamental class of Gromov-Witten theory.

Four-fermi Terms and Correlation Functions

- Put $h^1(s) = h^1(\mathcal{O}_{\mathbf{P}^1}(s))$. Then

$$F_\beta = \prod_c Q_c^{h^1(D_c \cdot \beta)},$$

where the product is taken over *all* linear equivalence classes c (i.e. without regard to any primitive collections).

- We have $c_1(X) \cdot \beta + \dim(X) + \deg(F_\beta) = \dim X_\beta$.
- If $P \in \mathrm{Sym}^k X$ with $k = c_1(X) \cdot \beta + \dim(X)$, then we can define the correlation function as

$$\langle P \rangle_\beta = [PF_\beta],$$

where the brackets denote the equivalence class of PF_β in the 1 dimensional vector space $\mathrm{Sym}^{\dim X_\beta} W / I_\beta$.

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Comparing sectors: $\mathbf{P}^1 \times \mathbf{P}^1$

- We already know a ring that surjects onto the quantum cohomology ring: $\mathrm{Sym}\, W$. We only have to identify the operator identities that they satisfy to identify the quantum cohomology ring
- Remaining problem: correlation functions from different sectors live in different vector spaces.
- If $\beta = (d, e)$ and $\beta' = (d', e')$ with $d' \geq d$ and $e' \geq e$, there is a natural map of polymologies

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which restricts to an isomorphism $f_{\beta'\beta}$ from $\mathrm{Sym}^{2d+2e+2} W/(Q^{d+1}, \tilde{Q}^{e+1})$ to $\mathrm{Sym}^{2d'+2e'+2} W/(Q^{d'+1}, \tilde{Q}^{e'+1})$.

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Comparing Sectors, Concluded

The maps $f_{\beta'\beta}$ are compatible and form a direct system. The direct limit is a one-dimensional vector space V containing all the correlation functions. In particular, correlation functions in different sectors can be compared.

Quantum Cohomology: $\mathbf{P}^1 \times \mathbf{P}^1$

We can now state the quantum cohomology relations for $\mathbf{P}^1 \times \mathbf{P}^1$:

$$Q = q_1, \qquad \tilde{Q} = q_2.$$

Here, the q_i are the GLSM version of the Kähler terms of the NLSM, depending on the FI parameters.

Verification of QC Relations for $\mathbf{P}^1 \times \mathbf{P}^1$

We have to show that for any $P \in \mathrm{Sym}^{2d+2e+2} W$ we have

$$\langle QP \rangle_{d+1,e} = \langle P \rangle_{d,e}$$

But since $f_{\beta'\beta}$ is multiplication by Q in this case, this amounts to the tautology $QP = Q(P)$.

General Case

- The general case is similar. We say that β' *dominates* β if the fan for $X_{\beta'}$ can be obtained from the fan for X_β by adding more edges to each linear equivalence class.
- Equivalent to $h^0(D_c \cdot \beta') \geq h^0(D_c \cdot \beta)$ for all c.
- These polymologies can be identified by multiplication by

$$\prod_c Q_c^{h^0(D_c \cdot \beta') - h^0(D_c \cdot \beta)},$$

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Batyrev's Relations

- We need a notion due to Batyrev. Let $v_\rho \in N$ be the primitive integral generator of $\rho \in \Sigma(1)$. Let K be a primitive collection.
- Consider $v_K = \sum_{\rho \in K} v_\rho$. Then there is a unique cone $\sigma \in \Sigma$ such that v_K is in the relative interior of σ.
- Batyrev shows that this gives a unique relation $\sum a_\rho v_\rho = 0$ with the properties $a_\rho = 1$ if $\rho \in K$, $a_\rho < 0$ if $\rho \in \sigma(1)$, and $a_\rho = 0$ otherwise.
- Furthermore, Batyrev shows that there is a unique $\beta_K \in H_2(X, \mathbf{Z})$ such that $D_\rho \cdot \beta_K = a_\rho$ for all $\rho \in \Sigma(1)$.

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- We can show that the edges of the cone σ in Batyrev's relation respects linear equivalence in the same way that K does. So we can partition $\sigma(1)$ into linear equivalence classes denoted T_K^-. Then the quantum cohomology relations are:
- $$\prod_{c \in T_K^+} Q_c = q^{\beta_K} \prod_{c \in T_K^-} Q_c^{-D_c \cdot \beta_K}.$$
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